# THE NORMAL SECTION CURVE

# ON AN ELLIPSOID

R. E. Deakin
School of Mathematical & Geospatial Sciences, RMIT University,
GPO Box 2476V, MELBOURNE VIC 3001, AUSTRALIA
email: rod.deakin@rmit.edu.au
November 2009

**ABSTRACT**

These notes provide a detailed derivation of the equation for a normal section curve on an ellipsoid and from this equation a technique for computing the arc length along a normal section curve is developed. Solutions for the direct and inverse problems of the normal section on an ellipsoid are given and MATLAB functions are provided showing the algorithms developed.

**INTRODUCTION**

In geodesy, the normal section curve is a plane curve created by intersecting a plane containing the normal to the ellipsoid (a normal section plane) with the surface of the ellipsoid, and the ellipsoid is a reference surface approximating the true shape of the Earth. In general, there are two normal section curves between two points on an ellipsoid, a fact that will be explained below, so the normal section curve is not a unique curve. And the distance along a normal section curve is not the shortest distance between two points. The shortest distance is along the geodesic, a unique curve on the surface defining the shortest distance, but the difference in length between the normal section and a geodesic can be shown to be negligible in all practical cases.

The azimuth of a normal section plane between two points on an ellipsoid can be easily determined by coordinate geometry if the latitudes and longitudes of the points are expressed in a local Cartesian coordinate system – this will be explained in detail below.

The distance along a normal section curve can be determined by numerical integration once the polar equation of the curve is known. And the derivation of the polar equation of a normal section curve is developed in detail by first proving that normal sections of ellipsoids are in fact ellipses, then deriving Cartesian equations of the ellipsoid and the normal section in local Cartesian coordinates and finally transforming the local Cartesian coordinates to polar coordinates. The differential equation for arc length (as a function of polar coordinates) is derived and a solution using a numerical technique known as Romberg integration is developed for the arc length along a normal section curve.

The azimuth of the normal section as a function of Cartesian coordinates); the polar equation of the normal section curve; and the solution of the arc length using Romberg integration are the core components of solutions of the direct and inverse cases of the normal sections on an ellipsoid. These are fundamental geodetic operations and can be likened to the equivalent operations of plane surveying; radiations (computing coordinates of points given bearings and distances radiating from a point of known coordinates) and joins; (computing bearings and distances between points having known coordinates). The solution of the direct and inverse cases of the normal section are set out in detail and MATLAB functions are provided.

**THE ELLIPSOID**

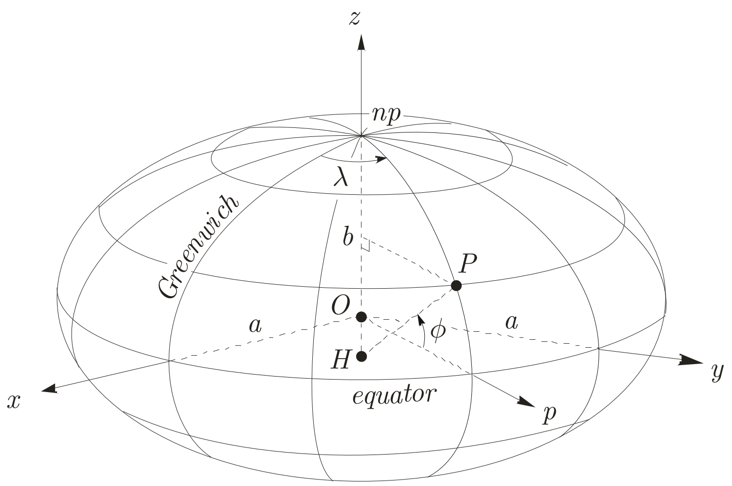



Figure 1: The reference ellipsoid

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse (whose major and minor semi-axes lengths are $a$ and $b$ respectively and $a > b$ ) about its minor axis. The $\phi, \lambda$ curvilinear coordinate system is a set of orthogonal parametric curves on the surface – parallels of latitude $\phi$ and meridians of longitude $\lambda$ with their respective reference planes; the equator and the Greenwich meridian.

Longitudes are measured $0^\circ$ to $\pm180^\circ$ (east positive, west negative) from the Greenwich meridian and latitudes are measured $0^\circ$ to $\pm90^\circ$ (north positive, south negative) from the equator. The $x,y,z$ geocentric Cartesian coordinate system has an origin at $O$, the centre of the ellipsoid, and the $z$-axis is the minor axis (axis of revolution). The $xOz$ plane is the Greenwich meridian plane (the origin of longitudes) and the $xOy$ plane is the equatorial plane.

The positive $x$-axis passes through the intersection of the Greenwich meridian and the equator, the positive $y$-axis is advanced $90^\circ$ east along the equator and the positive $z$-axis passes through the north pole of the ellipsoid.

The Cartesian equation of the ellipsoid is

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \tag{1}$$

where $a$ and $b$ are the semi-axes of the ellipsoid $(a>b)$.

The first-eccentricity squared $e^2$ and the flattening $f$ of the ellipsoid are defined by

$$\begin{aligned} e^2 &= \frac{a^2-b^2}{a^2} = f\left(2-f\right) \\ f &= \frac{a-b}{a} \end{aligned} \tag{2}$$

and the polar radius $c$, and the second-eccentricity squared $e'^2$ are defined by

$$\begin{aligned} c &= \frac{a^2}{b} = \frac{a}{1-f} \\ e'^2 &= \frac{a^2-b^2}{b^2} = \frac{f\left(2-f\right)}{\left(1-f\right)^2} = \frac{e^2}{1-e^2} \end{aligned} \tag{3}$$

## PROOF THAT NORMAL SECTION CURVES ARE ELLIPSES

Normal section curves are plane curves; i.e., curves on the surface of the ellipsoid created by intersecting the surface with a plane; and this plane (the normal section plane) contains the normal to the surface at one of the terminal points.

A meridian of longitude is also a normal section curve and <u>all meridians of longitude on the ellipsoid are ellipses</u> having semi-axes $a$ and $b$ $(a>b)$ since all meridian planes – e.g., Greenwich meridian plane $xOz$ and the meridian plane $pOz$ containing $P$ – contain the $z$-axis of the ellipsoid and their curves of intersection are ellipses (planes intersecting surfaces create curves of intersection on the surface). This can be seen if we let $p^2 = x^2 + y^2$ in equation (1) which gives the familiar equation of the (meridian) ellipse

$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1 \qquad (a < b) \tag{4}$$

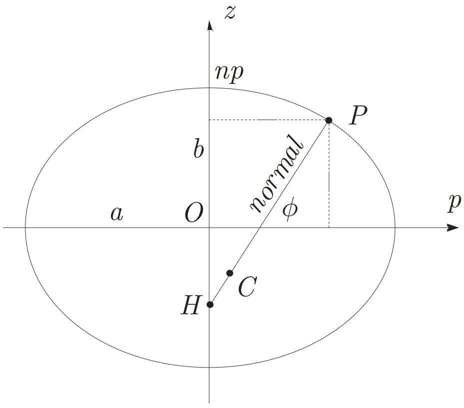



Figure 2: Meridian ellipse

In Figure 2, $\phi$ is the latitude of $P$ (the angle between the equator and the normal), $C$ is the centre of curvature and $PC$ is the radius of curvature of the meridian ellipse at $P$. $H$ is the intersection of the normal at $P$ and the $z$-axis (axis of revolution).

The only parallel of latitude that is also a normal section is the equator. And in this unique case, this normal section curve (the equator) is a circle. <u>All parallels of latitude on the ellipsoid are circles</u> created by intersecting the ellipsoid with planes parallel to (or coincident with) the $xOy$ equatorial plane. Replacing $z$ with a constant $C$ in equation (1) gives the equation for circular parallels of latitude

$$x^2 + y^2 = a^2\left(1 - \frac{C^2}{b^2}\right) = p^2 \qquad \left(0 \le C \le b; \;\; a > b\right) \tag{5}$$

<u>All other curves on the surface of the ellipsoid created by intersecting the ellipsoid with a plane are ellipses</u>. And this general statement covers all normal section planes that are not meridians or the equator. This can be demonstrated by using another set of coordinates $x', y', z'$ that are obtained by a rotation of the $x,y,z$ coordinates such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{where} \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where $\mathbf{R}$ is an orthogonal rotation matrix and $\mathbf{R}^{-1} = \mathbf{R}^T$ so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

giving

$$\begin{aligned} x^2 &= r_{11}^2 x'^2 + r_{21}^2 y'^2 + r_{31}^2 z'^2 + 2r_{11}r_{21}x'y' + 2r_{11}r_{31}x'z' + 2r_{21}r_{31}y'z' \\ y^2 &= r_{12}^2 x'^2 + r_{22}^2 y'^2 + r_{32}^2 z'^2 + 2r_{12}r_{22}x'y' + 2r_{12}r_{32}x'z' + 2r_{22}r_{32}y'z' \\ z^2 &= r_{13}^2 x'^2 + r_{23}^2 y'^2 + r_{33}^2 z'^2 + 2r_{13}r_{23}x'y' + 2r_{13}r_{33}x'z' + 2r_{23}r_{33}y'z' \\ x^2 + y^2 &= \left(r_{11}^2 + r_{12}^2\right)x'^2 + \left(r_{21}^2 + r_{22}^2\right)y'^2 + \left(r_{31}^2 + r_{32}^2\right)z'^2 + 2\left(r_{11}r_{21} + r_{12}r_{22}\right)x'y' \\ &\quad + 2\left(r_{11}r_{31} + r_{12}r_{32}\right)x'z' + 2\left(r_{21}r_{31} + r_{22}r_{32}\right)y'z' \end{aligned}$$

Substituting into equation (1) gives the equation of the ellipsoid in $x', y', z'$ coordinates

$$\begin{aligned} &\frac{1}{a^2}\begin{Bmatrix} \left(r_{11}^2 + r_{12}^2\right)x'^2 + \left(r_{21}^2 + r_{22}^2\right)y'^2 + \left(r_{31}^2 + r_{32}^2\right)z'^2 + 2\left(r_{11}r_{21} + r_{12}r_{22}\right)x'y' \\ +2\left(r_{11}r_{31} + r_{12}r_{32}\right)x'z' + 2\left(r_{21}r_{31} + r_{22}r_{32}\right)y'z' \end{Bmatrix} \\ &+ \frac{1}{b^2}\left\{r_{13}^2 x'^2 + r_{23}^2 y'^2 + r_{33}^2 z'^2 + 2r_{13}r_{23}x'y' + 2r_{13}r_{33}x'z' + 2r_{23}r_{33}y'z'\right\} = 1 \end{aligned} \tag{6}$$

In equation (6) let $z' = C_1$ where $C_1$ is a constant. The result will be the equation of a curve created by intersecting an inclined plane with the ellipsoid, i.e.,

$$\begin{aligned} &\left\{\frac{r_{11}^2 + r_{12}^2}{a^2} + \frac{r_{13}^2}{b^2}\right\}x'^2 + 2\left\{\frac{r_{11}r_{21} + r_{12}r_{22}}{a^2} + \frac{r_{13}r_{23}}{b^2}\right\}x'y' + \left\{\frac{r_{21}^2 + r_{22}^2}{a^2} + \frac{r_{23}^2}{b^2}\right\}y'^2 \\ &+ \left\{2C_1\left(r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33}\right)\right\}x' + \left\{2C_1\left(r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33}\right)\right\}y' \\ = {} & 1 - C_1^2\left\{r_{31}^2 + r_{32}^2 + r_{33}^2\right\} \end{aligned} \tag{7}$$

This equation can be expressed as

$$Ax'^2 + 2Hx'y' + By'^2 + Dx' + Ey' = 1 \tag{8}$$

where it can be shown that $AB - H^2 > 0$, hence it is the general Cartesian equation of an ellipse that is offset from the coordinate origin and rotated with respect to the coordinate axes (Grossman 1981). Equations of a similar form can be obtained for inclined planes $x' = C_2$ and $y' = C_3$, hence we may say, in general, inclined planes intersecting the ellipsoid will create curves of intersection that are ellipses.

## NORMAL SECTION CURVES BETWEEN $P_1$ AND $P_2$ ON THE ELLIPSOID

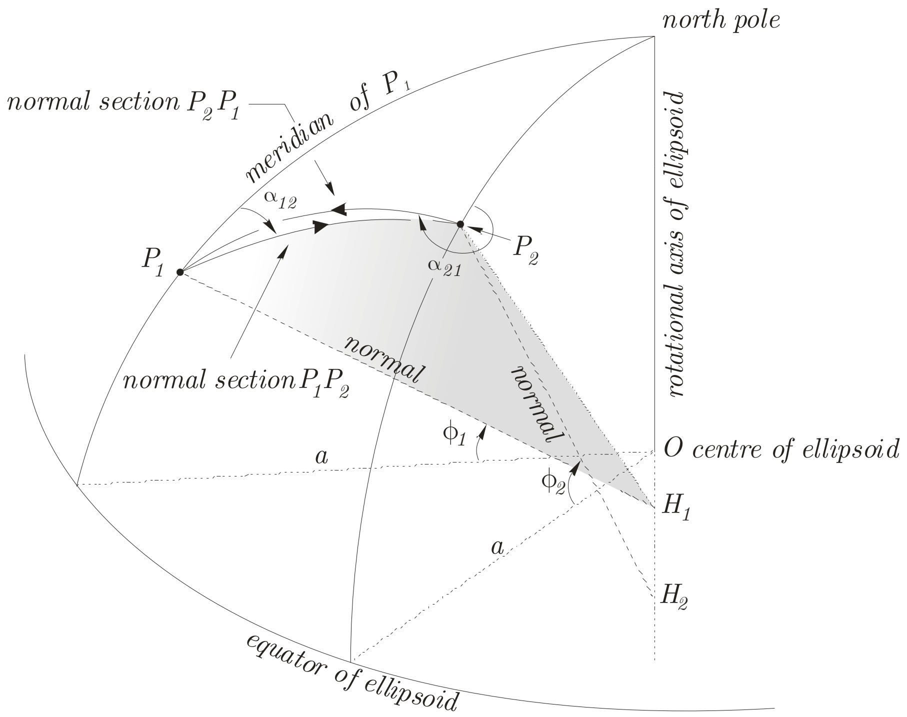



Figure 3: Normal section curves between $P_1$ and $P_2$ on the ellipsoid

Figure 3 shows $P_1$ and $P_2$ on the surface of an ellipsoid. The normals at $P_1$ and $P_2$ (that lie in the meridian planes $ONP_1H_1$ and $ONP_2H_2$ respectively) cut the rotational axis at $H_1$ and $H_2$, making angles $\phi_1, \phi_2$ with the equatorial plane of the ellipsoid. These are the latitudes of $P_1$ and $P_2$ respectively.

The plane containing the ellipsoid normal at $P_1$, and also the point $P_2$ intersects the surface of the ellipsoid along the normal section curve $P_1P_2$. The reciprocal normal section curve $P_2P_1$ (the intersection of the plane containing the normal at $P_2$, and also the point $P_1$ with the ellipsoidal surface) does not in general coincide with the normal section curve $P_1P_2$ although the distances along the two curves are, for all practical purposes, the same.

Hence there is not a unique normal section curve between $P_1$ and $P_2$, unless both $P_1$ and $P_2$ are on the same meridian or both are on the equator.

The azimuth $\alpha_{12}$, is the clockwise angle ($0^\circ$ to $360^\circ$) measured at $P_1$ in the local horizon plane from north (the direction of the meridian) to the normal section plane containing $P_2$. The azimuth $\alpha_{21}$ is the azimuth of the normal section plane $P_2P_1$ measured at $P_2$.

## LOCAL CARTESIAN COORDINATES

Figure 4 shows a local Cartesian coordinate system $E,N,U$ with an origin at $P$ on the reference ellipsoid with respect to the geocentric Cartesian system $x,y,z$ whose origin is a the centre of the ellipsoid

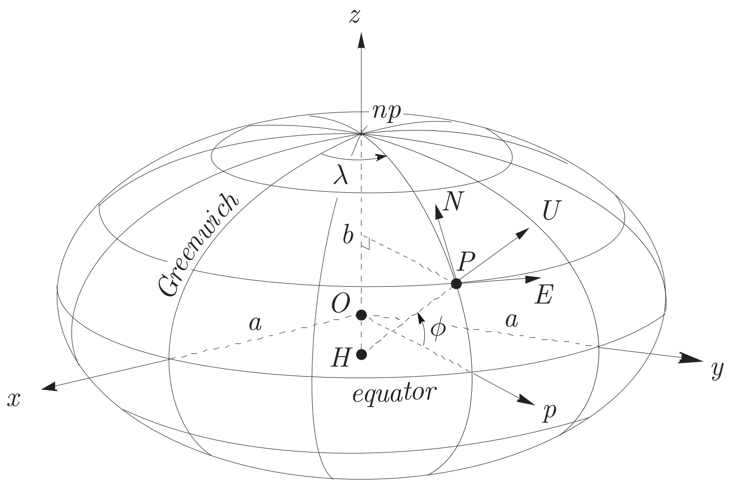



Figure 4: $x,y,z$ geocentric Cartesian and $E,N,U$ local Cartesian coordinates

Geocentric $x,y,z$ Cartesian coordinates are computed from the following equations

$$\begin{aligned} x &= \nu \cos\phi \cos\lambda \\ y &= \nu \cos\phi \sin\lambda \\ z &= \nu\left(1 - e^2\right)\sin\phi \end{aligned} \tag{9}$$

where $\nu = PH$ in Figure 4 is the radius of curvature in the prime vertical plane and

$$\nu = \frac{a}{\sqrt{1 - e^2 \sin^2\phi}} \tag{10}$$

The origin of the local $E,N,U$ system lies at the point $P\left(\phi_0, \lambda_0\right)$. The positive $U$-*axis* is coincident with the normal to the ellipsoid passing through $P$ and in the direction of increasing radius of curvature $\nu$. The $N$-$U$ plane lies in the meridian plane passing through $P$ and the positive $N$-axis points in the direction of North. The $E$-$U$ plane is perpendicular to the $N$-$U$ plane and the positive $E$-axis points East. The $E$-$N$ plane is often referred to as the local geodetic horizon plane.

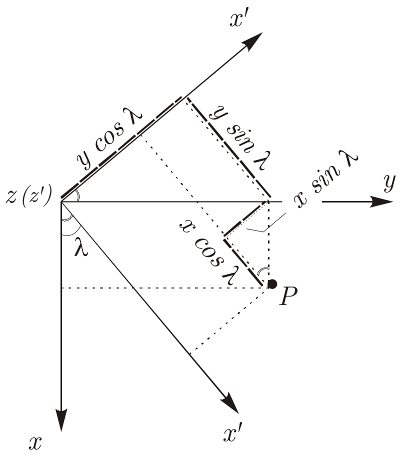
Geocentric and local Cartesian coordinates are related by the matrix equation

$$\begin{bmatrix} U \\ E \\ N \end{bmatrix} = \mathbf{R}_{\phi\lambda} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \tag{11}$$

where $x_0, x_0, z_0$ are the geocentric Cartesian coordinates of the origin of the $E, N, U$ system and $\mathbf{R}_{\phi\lambda}$ is a rotation matrix derived from the product of two separate rotation matrices.

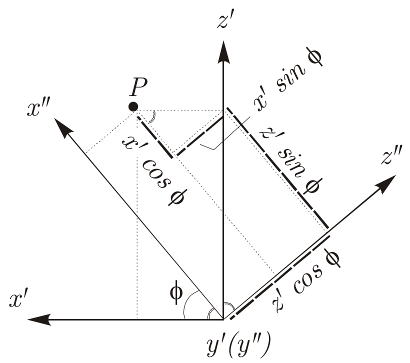
$$\mathbf{R}_{\phi\lambda} = \mathbf{R}_{\phi}\mathbf{R}_{\lambda} = \begin{bmatrix} \cos\phi_0 & 0 & \sin\phi_0 \\ 0 & 1 & 0 \\ -\sin\phi_0 & 0 & \cos\phi_0 \end{bmatrix} \begin{bmatrix} \cos\lambda_0 & \sin\lambda_0 & 0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

The first, $\mathbf{R}_{\lambda}$ (a positive right-handed rotation about the $x$-axis by $\lambda$) takes the $x,y,z$ axes to $x', y', z'$. The $z'$-axis is coincident with the *z-axis* and the $x'$-$y'$ plane is the Earth's equatorial plane. The $x'$-$y'$ plane is the meridian plane passing through $P$ and the $y'$-axis is perpendicular to the meridian plane and in the direction of East.




$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}_{\lambda}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The second $\mathbf{R}_{\phi}$ (a rotation about the $y'$-axis by $\phi$) takes the $x', y', z'$ axes to the $x'', y'', z''$ axes. The $x''$-axis is parallel to the $U$-axis, the $y''$-axis is parallel to the $E$-axis and the $z''$-axis is parallel to the $N$-axis.




$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}}_{\mathbf{R}_{\phi}} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Performing the matrix multiplication in equation (12) gives

$$\mathbf{R}_{\phi\lambda} = \begin{bmatrix} \cos\phi_0 \cos\lambda_0 & \cos\phi_0 \sin\lambda_0 & \sin\phi_0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0 \cos\lambda_0 & -\sin\phi_0 \sin\lambda_0 & \cos\phi_0 \end{bmatrix} \tag{13}$$

Rotation matrices formed from rotations about coordinate axes are often called Euler rotation matrices in honour of the Swiss mathematician Léonard Euler (1707-1783). They are orthogonal, satisfying the condition $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ (i.e., $\mathbf{R}^{-1} = \mathbf{R}^T$ ).

A re-ordering of the rows of the matrix $\mathbf{R}_{\phi\lambda}$ gives the transformation in the more usual form $E,N,U$

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \mathbf{R} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \tag{14}$$

where
$$\mathbf{R} = \begin{bmatrix} -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0 \cos\lambda_0 & -\sin\phi_0 \sin\lambda_0 & \cos\phi_0 \\ \cos\phi_0 \cos\lambda_0 & \cos\phi_0 \sin\lambda_0 & \sin\phi_0 \end{bmatrix} \tag{15}$$

From equation (14) we can see that coordinate differences $\Delta E = E_k - E_i$ , $\Delta N = N_k - N_i$ and $\Delta U = U_k - U_i$ in the local geodetic horizon plane are given by

$$\begin{bmatrix} \Delta E \\ \Delta N \\ \Delta U \end{bmatrix} = \mathbf{R} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{16}$$

where $\Delta x = x_k - x_i$ , $\Delta y = y_k - y_i$ and $\Delta z = z_k - z_i$ are geocentric Cartesian coordinate differences.

## NORMAL SECTION AZIMUTH ON THE ELLIPSOID

The matrix relationship given by equation (16) can be used to derive an expression for the azimuth of a normal section between two points on the reference ellipsoid. The normal section plane between points $P_1$ and $P_2$ on the Earth's terrestrial surface contains the normal at point $P_1$, the intersection of the normal and the rotational axis of the ellipsoid at $H_1$ (see Figure 3) and $P_2$. This plane will intersect the local geodetic horizon plane in a line having an angle with the north axis, which is the direction of the meridian at $P_1$.

This angle is the azimuth of the normal section plane $P_1 - P_2$ denoted as $\alpha_{12}$ and will have components $\Delta E$ and $\Delta N$ in the local geodetic horizon plane. From plane geometry

$$\tan \alpha_{12} = \frac{\Delta E}{\Delta N} \tag{17}$$

By inspection of equations (15) and (16) we may write the equation for normal section azimuth between points $P_1$ and $P_2$ as

$$\tan \alpha_{12} = \frac{\Delta E}{\Delta N} = \frac{-\Delta x \sin \lambda_1 + \Delta y \cos \lambda_1}{-\Delta x \sin \phi_1 \cos \lambda_1 - \Delta y \sin \phi_1 \sin \lambda_1 + \Delta z \cos \phi_1} \tag{18}$$

where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$ and $\Delta z = z_2 - z_1$

**EQUATION OF THE ELLIPSOID IN LOCAL CARTESIAN COORDINATES**

The Cartesian equation of the ellipsoid is given by equation (1) as

$$\boxed{\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1} \tag{19}$$

and multiplying both sides of equation (19) by $a^2$ gives

$$x^2 + y^2 + \frac{a^2}{b^2} z^2 = a^2 \tag{20}$$

Re-arranging equation (3) gives $\frac{a^2}{b^2} = e'^2 + 1$ and substituting this result into equation (20) and re-arranging gives an alternative expression for the Cartesian equation of an ellipsoid as

$$\boxed{x^2 + y^2 + z^2 + e'^2 z^2 - a^2 = 0} \tag{21}$$

We now find expressions for $x^2, y^2$ and $z^2$ in terms of local Cartesian coordinates that when substituted into equation (21) and simplified will give the equation of the ellipsoid in local Cartesian coordinates. The relevant substitutions are set out below.

The relationship between geocentric and local Cartesian coordinates is given by equation (14) as

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \mathbf{R} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \tag{22}$$

where the orthogonal rotation matrix **R** is given by equation (15) as

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0\cos\lambda_0 & -\sin\phi_0\sin\lambda_0 & \cos\phi_0 \\ \cos\phi_0\cos\lambda_0 & \cos\phi_0\sin\lambda_0 & \sin\phi_0 \end{bmatrix} \tag{23}$$

and

$$\begin{aligned} x_0 &= \nu_0\cos\phi_0\cos\lambda_0 \\ y_0 &= \nu_0\cos\phi_0\sin\lambda_0 \\ z_0 &= \nu_0\left(1-e^2\right)\sin\phi_0 \end{aligned} \tag{24}$$

with the radius of curvature of the prime vertical section

$$\nu_0 = \frac{a}{\sqrt{1-e^2\sin^2\phi_0}} \tag{25}$$

Re-arranging equation (22) gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}^{-1}\begin{bmatrix} E \\ N \\ U \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \tag{26}$$

where

$$\mathbf{R}^{-1} = \mathbf{R}^T = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \tag{27}$$

Expanding equation (26) gives

$$\begin{aligned} x &= r_{11}E + r_{21}N + r_{31}U + x_0 \\ y &= r_{12}E + r_{22}N + r_{32}U + y_0 \\ z &= r_{13}E + r_{23}N + r_{33}U + z_0 \end{aligned} \tag{28}$$

and

$$\begin{aligned} x^2 &= r_{11}^2E^2 + r_{21}^2N^2 + r_{31}^2U^2 + 2r_{11}r_{21}EN + 2r_{11}r_{31}EU + 2r_{21}r_{31}NU \\ &\quad + x_0^2 + 2r_{11}Ex_0 + 2r_{21}Nx_0 + 2r_{31}Ux_0 \\ y^2 &= r_{12}^2E^2 + r_{22}^2N^2 + r_{31}^2U^2 + 2r_{12}r_{22}EN + 2r_{12}r_{32}EU + 2r_{22}r_{32}NU \\ &\quad + y_0^2 + 2r_{12}Ey_0 + 2r_{22}Ny_0 + 2r_{32}Uy_0 \\ z^2 &= r_{13}^2E^2 + r_{23}^2N^2 + r_{33}^2U^2 + 2r_{13}r_{23}EN + 2r_{13}r_{33}EU + 2r_{23}r_{33}NU \\ &\quad + z_0^2 + 2r_{13}Ez_0 + 2r_{23}Nz_0 + 2r_{33}Uz_0 \end{aligned} \tag{29}$$

with

$$\begin{aligned}
x^2+y^2+z^2 = {} & \left(r_{11}^2+r_{12}^2+r_{13}^2\right)E^2+\left(r_{21}^2+r_{22}^2+r_{23}^2\right)N^2+\left(r_{31}^2+r_{32}^2+r_{33}^2\right)U^2 \\
& +2\left(r_{11}r_{21}+r_{12}r_{22}+r_{13}r_{23}\right)EN \\
& +2\left(r_{11}r_{31}+r_{12}r_{32}+r_{13}r_{33}\right)EU \\
& +2\left(r_{21}r_{31}+r_{22}r_{32}+r_{23}r_{33}\right)NU \\
& +x_0^2+y_0^2+z_0^2 \\
& +2\left(r_{11}x_0+r_{12}y_0+r_{13}z_0\right)E \\
& +2\left(r_{21}x_0+r_{22}y_0+r_{23}z_0\right)N \\
& +2\left(r_{31}x_0+r_{32}y_0+r_{33}z_0\right)U
\end{aligned} \tag{30}$$

Now using the equivalences for $r_{11}, r_{12}, \text{etc}$ given in equation (23), certain terms in equation (30) can be simplified as

$$\begin{aligned}
r_{11}^2+r_{12}^2+r_{13}^2 &= \sin^2\lambda_0+\cos^2\lambda_0=1 \\
r_{21}^2+r_{22}^2+r_{23}^2 &= \sin^2\phi_0\left(\cos^2\lambda_0+\sin^2\lambda_0\right)+\cos^2\phi_0=1 \\
r_{31}^2+r_{32}^2+r_{33}^2 &= \cos^2\phi_0\left(\cos^2\lambda_0+\sin^2\lambda_0\right)+\sin^2\phi_0=1
\end{aligned}$$

and

$$\begin{aligned}
r_{11}r_{21}+r_{12}r_{22}+r_{13}r_{23} &= \sin\lambda_0\sin\phi_0\cos\lambda_0-\cos\lambda_0\sin\phi_0\sin\lambda_0+0 \\
&= 0 \\
r_{11}r_{31}+r_{12}r_{32}+r_{13}r_{33} &= -\sin\lambda_0\cos\phi_0\cos\lambda_0+\cos\lambda_0\cos\phi_0\sin\lambda_0+0 \\
&= 0 \\
r_{21}r_{31}+r_{22}r_{32}+r_{23}r_{33} &= -\sin\phi_0\cos\phi_0\cos^2\lambda_0-\sin\phi_0\cos\phi_0\sin^2\lambda_0+\cos\phi_0\sin\phi_0 \\
&= -\sin\phi_0\cos\phi_0\left(\cos^2\lambda_0+\sin^2\lambda_0\right)+\cos\phi_0\sin\phi_0 \\
&= 0
\end{aligned}$$

Substituting these results into equation (30) gives

$$\begin{aligned}
x^2+y^2+z^2 = {} & E^2+N^2+U^2+x_0^2+y_0^2+z_0^2 \\
& +2\left(r_{11}x_0+r_{12}y_0+r_{13}z_0\right)E \\
& +2\left(r_{21}x_0+r_{22}y_0+r_{23}z_0\right)N \\
& +2\left(r_{31}x_0+r_{32}y_0+r_{33}z_0\right)U
\end{aligned} \tag{31}$$

Using equation (24) and noting that equation (25) can be re-arranged as $1-e^2\sin^2\phi_0=\dfrac{a^2}{\nu_0^2}$ we have

$$
\begin{aligned}
x_0^2 + y_0^2 + z_0^2 &= \nu_0^2 \cos^2\phi_0 \left(\cos^2\lambda_0 + \sin^2\lambda_0\right) + \nu_0^2 \left(1 - e^2\right)^2 \sin^2\phi_0 \\
&= \nu_0^2 \cos^2\phi_0 + \nu_0^2 \sin^2\phi_0 \left(1 - 2e^2 + e^4\right) \\
&= \nu_0^2 \cos^2\phi_0 + \nu_0^2 \sin^2\phi_0 - 2\nu_0^2 e^2 \sin^2\phi_0 + \nu_0^2 e^4 \sin^2\phi_0 \\
&= \nu_0^2 - 2\nu_0^2 e^2 \sin^2\phi_0 + \nu_0^2 e^4 \sin^2\phi_0 \\
&= \nu_0^2 \left(1 - e^2 \sin^2\phi_0\right) - \nu_0^2 e^2 \sin^2\phi_0 \left(1 - e^2\right) \\
&= a^2 - \left(\nu_0^2 - a^2\right)\left(1 - e^2\right)
\end{aligned}
$$

From equations (31), (23) and (24) we have

$$
\begin{aligned}
r_{11}x_0 + r_{12}y_0 + r_{13}z_0 &= -\nu_0 \cos\phi_0 \cos\lambda_0 \sin\lambda_0 + \nu_0 \cos\phi_0 \sin\lambda_0 \cos\lambda_0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
r_{21}x_0 + r_{22}y_0 + r_{23}z_0 &= -\nu_0 \cos\phi_0 \sin\phi_0 \cos^2\lambda_0 - \nu_0 \sin\phi_0 \cos\phi_0 \sin^2\lambda_0 \\
&\quad + \nu_0 \left(1 - e^2\right) \sin\phi_0 \cos\phi_0 \\
&= -\nu_0 \cos\phi_0 \sin\phi_0 \left(\cos^2\lambda_0 + \sin^2\lambda_0 - 1 + e^2\right) \\
&= -\nu_0 e^2 \cos\phi_0 \sin\phi_0
\end{aligned}
$$

and

$$
\begin{aligned}
r_{31}x_0 + r_{32}y_0 + r_{33}z_0 &= \nu_0 \cos^2\phi_0 \cos^2\lambda_0 + \nu_0 \cos^2\phi_0 \sin^2\lambda_0 + \nu_0 \left(1 - e^2\right) \sin^2\phi_0 \\
&= \nu_0 \cos^2\phi_0 + \nu_0 \left(1 - e^2\right) \sin^2\phi_0 \\
&= \nu_0 \cos^2\phi_0 + \nu_0 \sin^2\phi_0 - \nu_0 e^2 \sin^2\phi_0 \\
&= \nu_0 \left(1 - e^2 \sin^2\phi_0\right) \\
&= a^2
\end{aligned}
$$

Substituting these results into equation (31) gives

$$
\begin{aligned}
x^2 + y^2 + z^2 = {} & E^2 + N^2 + U^2 + \nu_0^2 \left(1 - e^2 \sin^2\phi_0\right) - \nu_0^2 e^2 \sin^2\phi_0 \left(1 - e^2\right) \\
& - 2\nu_0 e^2 \sin\phi_0 \cos\phi_0 N + 2\nu_0 \left(1 - e^2 \sin^2\phi_0\right) U
\end{aligned}
\tag{32}
$$

Using the expression for $z^2$ given in equation (29), the term $e'^2 z^2$ in equation (21) can be expressed as

$$
\begin{aligned}
e'^2 z^2 = e'^2 \Big\{ & r_{13}^2 E^2 + r_{23}^2 N^2 + r_{33}^2 U^2 + 2r_{13}r_{23}EN + 2r_{13}r_{33}EU + 2r_{23}r_{33}NU \\
& + z_0^2 + 2r_{13}Ez_0 + 2r_{23}Nz_0 + 2r_{33}Uz_0 \Big\}
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
& r_{13}^2 = 0; \quad r_{23}^2 = \cos^2\phi; \quad r_{33}^2 = \sin^2\phi; \\
& 2r_{13}r_{23} = 0; \quad 2r_{13}r_{33} = 0; \quad 2r_{23}r_{33} = 2\cos\phi_0 \sin\phi_0; \\
& z_0^2 = \nu_0^2 \left(1 - e^2\right)^2 \sin^2\phi_0; \\
& 2r_{13}z_0 = 0; \quad 2r_{23}z_0 = 2\nu_0 \left(1 - e^2\right) \cos\phi_0 \sin\phi_0; \quad 2r_{33}z_0 = 2\nu_0 \left(1 - e^2\right) \sin^2\phi_0
\end{aligned}
$$

and equation (33) can be expressed as

$$
\begin{aligned}
e'^2 z^2 &= e'^2 \left(\cos^2\phi_0 N^2 + \sin^2\phi_0 U^2 + 2\cos\phi_0 \sin\phi_0 NU\right) \\
&\quad + e'^2 \left(\nu_0^2 \left(1-e^2\right)^2 \sin^2\phi_0 + 2\nu_0 \left(1-e^2\right)\cos\phi_0 \sin\phi_0 N + 2\nu_0 \left(1-e^2\right)\sin^2\phi_0 U\right)
\end{aligned}
$$

But $e'^2 = \dfrac{e^2}{1-e^2}$ so we may write

$$
\begin{aligned}
e'^2 z^2 &= e'^2 \left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2 \\
&\quad + \frac{e^2}{1-e^2}\left(\nu_0^2 \left(1-e^2\right)^2 \sin^2\phi_0 + 2\nu_0 \left(1-e^2\right)\cos\phi_0 \sin\phi_0 N + 2\nu_0 \left(1-e^2\right)\sin^2\phi_0 U\right) \\
&= e'^2 \left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2 \\
&\quad + \nu_0^2 \left(1-e^2\right)^2 e^2 \sin^2\phi_0 + 2\nu_0 e^2 \cos\phi_0 \sin\phi_0 N + 2\nu_0 e^2 \sin^2\phi_0 U
\end{aligned}
\tag{34}
$$

Substituting equations (32) and (34) into equation (21) gives

$$
\begin{aligned}
E^2 + N^2 + U^2 + e'^2 &\left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2 - a^2 \\
&+ \nu_0^2 \left(1 - e^2 \sin^2\phi_0\right) - \nu_0^2 e^2 \sin^2\phi_0 \left(1-e^2\right) \\
&- 2\nu_0 e^2 \sin\phi_0 \cos\phi_0\, N + 2\nu_0 \left(1 - e^2 \sin^2\phi_0\right) U \\
&+ \nu_0^2 e^2 \sin^2\phi_0 \left(1-e^2\right) + 2\nu_0 e^2 \sin\phi_0 \cos\phi_0\, N + 2\nu_0 e^2 \sin^2\phi_0\, U = 0
\end{aligned}
$$

And simplifying and noting that $\nu_0^2 \left(1 - e^2 \sin^2\phi_0\right) = a^2$ gives the Cartesian equation of the ellipsoid in local coordinates $E, N, U$ as

$$
\boxed{E^2 + N^2 + U^2 + e'^2 \left(\cos\phi_0\, N + \sin\phi_0\, U\right)^2 + 2\nu_0 U = 0} \tag{35}
$$

The origin of the $E, N, U$ system is at $P_1$ with coordinates $\phi_0, \lambda_0$ where the radius of curvature of the prime vertical section is $\nu_0 = \dfrac{a}{\left(1 - e^2 \sin^2\phi_0\right)^{\frac{1}{2}}}$ and the first and second eccentricities of the ellipsoid $\left(a, f\right)$ are obtained from $e^2 = f\left(2 - f\right)$ and $e'^2 = \dfrac{e^2}{1-e^2}$

Equation (35) is similar to an equation given by Bowring (1978, p. 363, equation (10) with $x \equiv N \quad y \equiv -U$ , $z \equiv E$ ). Bowring does not give a derivation, but notes that his equation is taken from Tobey (1928).

## CARTESIAN EQUATION OF THE NORMAL SECTION CURVE

The Cartesian equation of the normal section curve is developed as a function of local Cartesian coordinates $\zeta, \eta, \xi$ which are rotated from the local $E,N,U$ system by the azimuth $\alpha$ of the normal section plane.

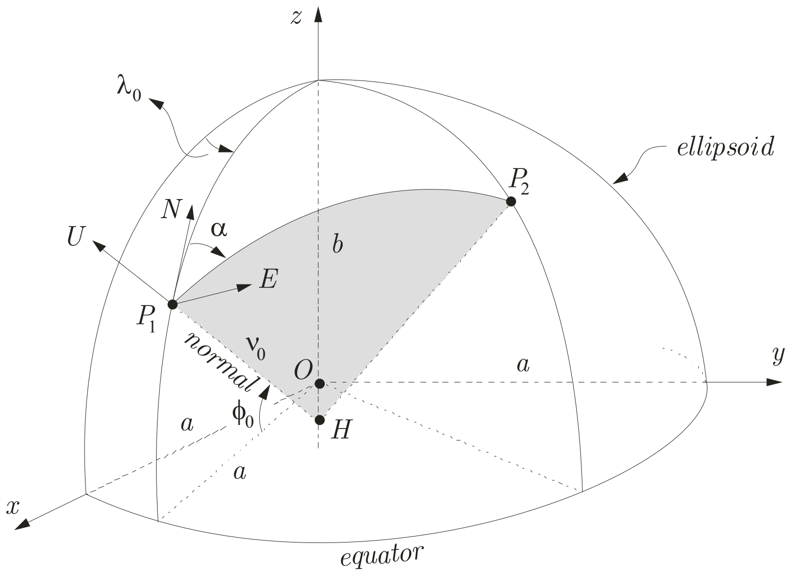



Figure 5: Normal section plane between $P_1$ and $P_2$ on the ellipsoid

Figure 5 shows a normal section plane having an azimuth $\alpha$ between $P_1$ and $P_2$ on the ellipsoid and a local Cartesian coordinate system $E,N,U$ with an origin at $P_1$.

Cartesian equations of the ellipsoid in geocentric and local coordinates given by equations (1), (21) and (35) are:

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$$

$$x^2+y^2+z^2+e'^2z^2-a^2=0$$

$$E^2+N^2+U^2+e'^2\left(\cos\phi_0\,N+\sin\phi_0\,U\right)^2+2\nu_0U=0$$

Consider a rotation of the $E,N,U$ system about the $U$-axis by the azimuth $\alpha$ so that the rotated $N$-axis lies in the normal section plane and the rotated $E$-axis is perpendicular to the plane. Denote this rotated $E,N,U$ system as $\zeta, \eta, \xi$ shown in Figure 6

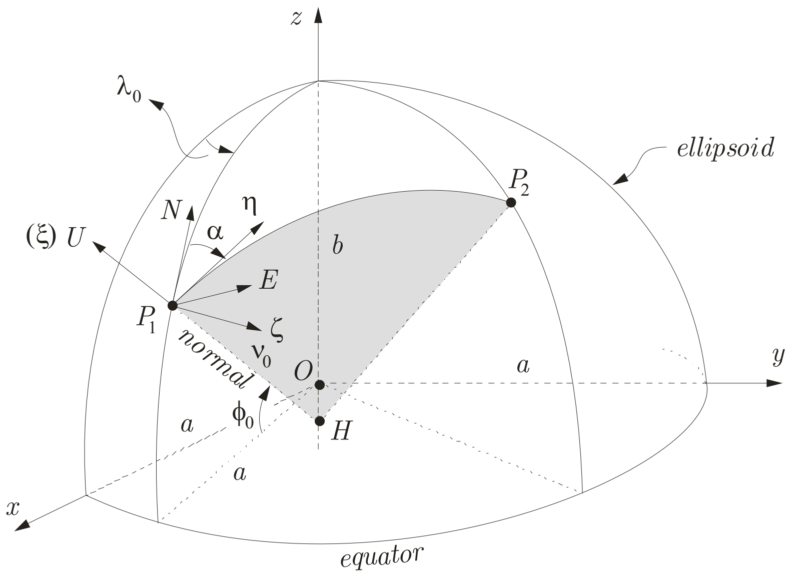
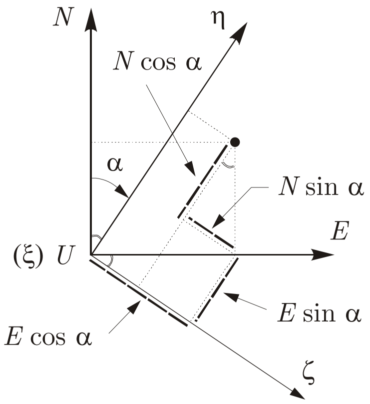



Figure 6: Rotated local coordinate system $\zeta, \eta, \xi$

These two local Cartesian systems; $E, N, U$ and $\zeta, \eta, \xi$ are related by




$$\begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ U \end{bmatrix}$$

and

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix}$$

and we may write

$$\begin{aligned} E &= \zeta \cos\alpha + \eta \sin\alpha ; & E^2 &= \zeta^2 \cos^2\alpha + \eta^2 \sin^2\alpha + 2\zeta\eta \cos\alpha \sin\alpha \\ N &= \eta \cos\alpha - \zeta \sin\alpha ; & N^2 &= \zeta^2 \sin^2\alpha + \eta^2 \cos^2\alpha - 2\zeta\eta \cos\alpha \sin\alpha \\ U &= \xi & U^2 &= \xi^2 \end{aligned} \tag{36}$$

giving

$$E^2 + N^2 + U^2 = \zeta^2 + \eta^2 + \xi^2 \tag{37}$$

Substituting equations (36) and (37) into equation (35) gives

$$\zeta^2 + \eta^2 + \xi^2 + e'^2\left(-\zeta\sin\alpha\cos\phi_0 + \eta\cos\alpha\cos\phi_0 + \xi\sin\phi_0\right)^2 + 2\nu_0\xi = 0 \tag{38}$$

This is the Cartesian equation of an ellipsoid where the local Cartesian coordinates $\zeta, \eta, \xi$ have an origin at $P_1\left(\phi_0, \lambda_0\right)$ on the ellipsoid $\left(a, f\right)$ with the $\xi$-axis in the direction of the outward normal at $P_1$; the $\xi$-$\eta$ plane is coincident with the normal section plane making an angle $\alpha$ with the meridian plane of $P_1$; and the $\xi$-$\zeta$ plane is perpendicular to the normal section plane. As before the radius of curvature of the prime vertical section is $\nu_0 = \dfrac{a}{\left(1 - e^2\sin^2\phi_0\right)^{\frac{1}{2}}}$ and the first and second eccentricities of the ellipsoid are obtained from $e^2 = f\left(2 - f\right)$ and $e'^2 = \dfrac{e^2}{1 - e^2}$.

Setting $\zeta = 0$ in equation (38) will give the equation of the normal section plane as

$$\eta^2 + \xi^2 + e'^2\left(\eta\cos\alpha\cos\phi_0 + \xi\sin\phi_0\right)^2 + 2\nu_0\xi = 0 \tag{39}$$

Expanding equation (39) gives

$$\eta^2 + \eta^2 e'^2\cos^2\alpha\cos^2\phi_0 + \xi^2 + \xi^2 e'^2\sin^2\phi_0 + 2\eta\xi e'^2\cos\alpha\cos\phi_0\sin\phi_0 + 2\nu_0\xi = 0$$

which can be simplified to

$$\xi^2\left(1 + g^2\right) + 2\,\xi\eta gh + \eta^2\left(1 + h^2\right) + 2\nu_0\xi = 0 \tag{40}$$

where $g$ and $h$ are <u>constants of the normal section</u> and

$$\begin{aligned} g &= e'\sin\phi_0 &&= \frac{e}{\sqrt{1 - e^2}}\sin\phi_0 \\ h &= e'\cos\alpha\cos\phi_0 &&= \frac{e}{\sqrt{1 - e^2}}\cos\alpha\sin\phi_0 \end{aligned} \tag{41}$$

Equation (40) is similar to Clarke (1880, equation 14, p. 107) although Clarke's derivation is different and very concise; taking only 11 lines of text and diagrams.

## POLAR EQUATION OF THE NORMAL SECTION CURVE

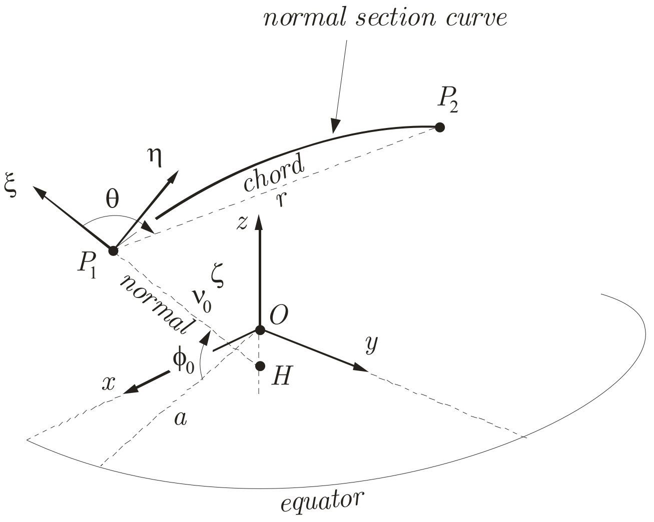



Figure 7: Normal section curve $f\left(\xi,\eta\right)$

The Cartesian equation of the normal section curve in local coordinates $\xi,\eta,\zeta=0$ is given by equations (40) and (41) given the latitude $\phi_0$ of $P_1$, the ellipsoid constant $e^2$ and the azimuth $\alpha$ of the normal section plane.

The equation of the curve in <u>polar coordinates</u> $r,\theta$; where $r$ is a chord of the curve and $\theta$ is the zenith distance of the chord, can be obtained in the following manner.

First, from Figure 7, we may write

$$\begin{aligned}\xi &= r\cos\theta \\ \eta &= r\sin\theta\end{aligned} \tag{42}$$

And second, we may re-arrange equation (40) as

$$\xi^2+\eta^2+\left(g\xi+h\eta\right)^2=-2\nu_0\xi \tag{43}$$

Squaring equations (42) and adding gives

$$\xi^2+\eta^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2 \tag{44}$$

and the third term in equation (43) can be expressed as

$$\begin{aligned}\left(g\xi+h\eta\right)^2 &= \left(g\,r\cos\theta+h\,r\sin\theta\right)^2 \\ &= g^2r^2\cos^2\theta+h^2r^2\sin^2\theta+2gh\,r^2\sin\theta\cos\theta \\ &= r^2\left(g\cos\theta+h\sin\theta\right)^2\end{aligned} \tag{45}$$

Substituting equations (44) and (45) into equation (43) and re-arranging gives the polar equation of the normal section curve

$$\boxed{r = \frac{-2\nu_0 \cos\theta}{1 + \left(g\cos\theta + h\sin\theta\right)^2}} \tag{46}$$

**ARC LENGTH ALONG A NORMAL SECTION CURVE**

To evaluate the arc length $s$ along the normal section curve, consider the following

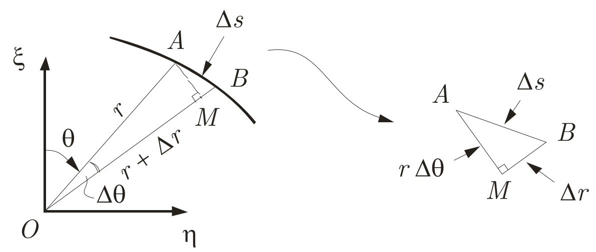



Figure 8: Small element of arc length along a normal section curve

In Figure 8, when $\Delta\theta$ is small, then $AM \simeq r\,\Delta\theta$ and the arc length $\Delta s$ is approximated by the chord $AB$ and $\left(\Delta s\right)^2 \simeq \left(r\,\Delta\theta\right)^2 + \left(\Delta r\right)^2$ or

$$\begin{aligned}\Delta s &= \sqrt{\left(r\,\Delta\theta\right)^2 + \left(\Delta r\right)^2} \\ &= \sqrt{\left(\Delta\theta\right)^2\left(r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2\right)}\end{aligned}$$

and

$$\frac{\Delta s}{\Delta\theta} = \sqrt{r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2}$$

Taking the limit of $\dfrac{\Delta s}{\Delta\theta}$ as $\Delta\theta \to 0$ gives

$$\lim_{\Delta\theta\to 0}\left(\frac{\Delta s}{\Delta\theta}\right) = \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \tag{47}$$

and the arc length is given by

$$s = \int ds = \int_{\theta=\theta_A}^{\theta=\theta_B} \left\{ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta \tag{48}$$

Referring to Figure 7 the $\eta$-axis is tangential to the normal section curve $P_1P_2$ at $P_1$ and the zenith distance $\theta = \theta_A = \dfrac{\pi}{2}$ and $r = 0$. And when $\theta = \theta_B = \theta_2$ then the chord $r = P_1P_2$ and the arc length of the normal section curve is given by

$$\boxed{s = \int ds = \int_{\theta=\frac{\pi}{2}}^{\theta=\theta_2} \left\{ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta} \tag{49}$$

$r$ is given by equation (46) with normal section constants $g$ and $h$ given by equations (41).

The derivative $\dfrac{dr}{d\theta}$ can be obtained from equation (46) using the quotient rule for differential calculus

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left( \frac{u}{v} \right) = \frac{v\dfrac{du}{d\theta} - u\dfrac{dv}{d\theta}}{v^2} \tag{50}$$

where

$$\begin{aligned} u &= -2\nu_0 \cos\theta; & v &= 1 + \left( g\cos\theta + h\sin\theta \right)^2 \\ \frac{du}{d\theta} &= 2\nu_0 \sin\theta; & \frac{dv}{d\theta} &= 2\left( g\cos\theta + h\sin\theta \right)\left( h\cos\theta - g\sin\theta \right) \end{aligned} \tag{51}$$

The arc length of the normal section curve between $P_1$ and $P_2$ can be found by evaluating the integral given in equation (49). This integral cannot be solved analytically but may be evaluated by a numerical integration technique known Romberg integration. Appendix 1 contains a development of the formula used in Romberg integration as well as a MATLAB function demonstrating the algorithm.

## SOLVING THE DIRECT AND INVERSE PROBLEMS ON THE ELLIPSOID USING NORMAL SECTIONS

The direct problem on an ellipsoid is: given latitude and longitude of $P_1$, azimuth $\alpha_{12}$ of the normal section $P_1P_2$ and the arc length $s$ along the normal section curve; compute the latitude and longitude of $P_2$.

The inverse problem on an ellipsoid is: given the latitudes and longitudes of $P_1$ and $P_2$ compute the azimuth $\alpha_{12}$ and the arc length $s$ along the normal section curve $P_1P_2$.

Note 1. In general there are two normal section curves joining $P_1$ and $P_2$. We are only dealing with the single normal section $P_1P_2$ (containing the normal at $P_1$ – see Figure 3) and so only the forward azimuth $\alpha_{12}$ is given or computed. The reverse azimuth $\alpha_{21}$ is the azimuth of the normal section $P_2P_1$ (containing the normal at $P_2$) which is a different curve from normal section curve $P_1P_2$.

Note 2. The usual meaning of: *solving the direct and inverse problems on the ellipsoid* would imply the use of the geodesic; the unique curve defining the shortest distance between two points. And solving these problems is usually done using Bessel's method with Vincenty's equations (Deakin & Hunter 2007) or Pittman's method (Deakin & Hunter 2007).

In the solutions of the direct and inverse problems set out in subsequent sections, the following notation and relationships are used.

- $a, f$ semi-major axis length and flattening of ellipsoid.
- $b$ semi-minor axis length of the ellipsoid, $b = a(1-f)$
- $e^2$ eccentricity of ellipsoid squared, $e^2 = f(2-f)$
- $e'^2$ 2nd-eccentricity of ellipsoid squared, $e'^2 = \dfrac{e^2}{1-e^2}$
- $\phi, \lambda$ latitude and longitude on ellipsoid: $\phi$ measured $0^\circ$ to $\pm 90^\circ$ (north latitudes positive and south latitudes negative) and $\lambda$ measured $0^\circ$ to $\pm 180^\circ$ (east longitudes positive and west longitudes negative).
- $s$ length of the normal section curve on the ellipsoid.
- $\alpha_{12}$ azimuth of normal section $P_1P_2$
- $\alpha'_{12}$ azimuth of normal section $P_2P_1$ (measured in the local horizon plane of $P_1$)
- $\alpha_{21}$ reverse azimuth; azimuth of normal section $P_2P_1$
- $c$ chord $P_1P_2$
- $\theta$ zenith distance of the chord $c$

$x,y,z$ are geocentric Cartesian coordinates with an origin at the centre of the ellipsoid and where the $z$-axis is coincident with the rotational axis of the ellipsoid, the $x$-$z$ plane is the Greenwich meridian plane and the $x$-$y$ plane is the equatorial plane of the ellipsoid.

$x',y',z'$ are geocentric Cartesian coordinates with an origin at the centre of the ellipsoid and where the $z'$-axis is coincident with the rotational axis of the ellipsoid, the $x'$-$z'$ plane is the meridian plane of $P_1$ and the $x'$-$y'$ plane is the equatorial plane of the ellipsoid. The $x',y',z'$ system is rotated from the $x,y,z$ system by an angle $\lambda_1$ about the $z$-axis.

vectors a vector **a** defining the length and direction of a line from point 1 to point 2 is given by the formula $\mathbf{a} = a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}$ where $a_i = x_2 - x_1$, $a_j = y_2 - y_1$ and $a_k = z_2 - z_1$ are the vector components and **i**, **j**, and **k** are unit vectors in the direction of the positive $x$, $y$, and $z$ axes respectively. The components of a unit vector $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ can be calculated by dividing each component by the magnitude of the vector $|\mathbf{a}| = \sqrt{a_i^2 + a_j^2 + a_k^2}$.

For vectors **a** and **b** the <u>vector dot product</u> is $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where $\theta$ is the angle between the vectors. For unit vectors $\hat{\mathbf{a}} \bullet \hat{\mathbf{b}} = \cos\theta$. The vector dot product is a scalar quantity $S = a_ib_i + a_jb_j + a_kb_k$, hence for unit vectors the angle between them is given by $\cos\theta = S$.

For vectors **a** and **b** the <u>vector cross product</u> is $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{p}}$ where $\hat{\mathbf{p}}$ is a unit vector perpendicular to the plane containing **a** and **b** and in the direction of a right-handed screw rotated from **a** to **b**. The result of a vector cross product is another vector whose components are given by $\mathbf{a} \times \mathbf{b} = \left(a_jb_k - a_kb_j\right)\mathbf{i} - \left(a_ib_k - a_kb_i\right)\mathbf{j} + \left(a_ib_j - a_jb_i\right)\mathbf{k}$. The components of the unit vector $\hat{\mathbf{p}}$ are found by dividing each component of the cross product by the magnitudes $|\mathbf{a}|$ and $|\mathbf{b}|$, and the sine of the angle between them. For unit vectors $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \sin\theta\,\hat{\mathbf{p}}$ and for perpendicular unit vectors $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{p}}$.

## THE DIRECT PROBLEM ON THE ELLIPSOID USING A NORMAL SECTION

The direct problem is: Given latitude and longitude of $P_1$, azimuth $\alpha_{12}$ of the normal section $P_1P_2$ and the arc length $s$ along the normal section curve; compute the latitude and longitude of $P_2$.

With the ellipsoid constants $a$, $f$, $e^2$ and $e'^2$ and given $\phi_1, \lambda_1, \alpha_{12}$ and $s$ the problem may be solved by the following sequence.

1. Compute $\nu_1$ the radius of curvature in the prime vertical plane of $P_1$ from

$$\nu_1 = \frac{a}{\left(1 - e^2 \sin^2 \phi_1\right)^{\frac{1}{2}}}$$

2. Compute the constants $g$ and $h$ of the normal section $P_1P_2$ from

$$\begin{aligned} g &= e' \sin\phi_1 &&= \frac{e}{\sqrt{1-e^2}} \sin\phi_1 \\ h &= e' \cos\alpha_{12} \cos\phi_1 &&= \frac{e}{\sqrt{1-e^2}} \cos\alpha_{12} \sin\phi_1 \end{aligned}$$

3. Compute the chord $c = P_1P_2$ and the zenith distance $\theta$ of the chord $P_1P_2$ by iteration using the following sequence of operations until there is negligible change in the computed chord distance

   start  Set the counter $k = 1$ and set the chord $c_k = s$

   (i) Set the counter $n = 1$ and set the zenith distance $\theta_n = \dfrac{\pi}{2}$

   (ii) Use Newton-Raphson iteration to compute the zenith distance of the chord using equation (46) rearranged as

   $f\left(\theta\right) = c + c\left(g\cos\theta + h\sin\theta\right)^2 - 2\nu_1 \cos\theta = 0$ and the iterative formula

   $\theta_{n+1} = \theta_n - \dfrac{f\left(\theta_n\right)}{f'\left(\theta_n\right)}$ where $f'\left(\theta_n\right)$ is the derivative of $f\left(\theta_n\right)$ and

$$\begin{aligned} f\left(\theta_n\right) &= c_k + c_k\left(g\cos\theta_n + h\sin\theta_n\right)^2 - 2\nu_1\cos\theta_n \\ f'\left(\theta_n\right) &= 2c_k\left(g\cos\theta_n + h\sin\theta_n\right)\left(h\cos\theta_n - g\sin\theta_n\right) - 2\nu_1\sin\theta_n \end{aligned}$$

   Note that the iteration for $\theta$ is terminated when $\theta_n$ and $\theta_{n+1}$ differ by an acceptably small value.

   (iii) Compute the arc length $s_k$ using Romberg integration given $a, f, \phi_1, \alpha_{12}, \theta$

(iv) Compute the small change in arc length $ds = s_k - s$

(v) If $ds < 0.000001$ then go to end; else go (vi)

(vi) Increment $k$, compute new chord $c_k = c_{k-1} - ds$ and go to (i)

end Iteration for the chord $c = P_1P_2$ and the zenith distance $\theta$ of the chord $P_1P_2$ is complete.

4. Compute the $x,y,z$ coordinates of $P_1$ using

$$\begin{aligned} x_1 &= \nu_1 \cos\phi_1 \cos\lambda_1 \\ y_1 &= \nu_1 \cos\phi_1 \sin\lambda_1 \\ z_1 &= \nu_1 \left(1 - e^2\right)\sin\phi_1 \end{aligned}$$

5. Compute coordinate differences $\Delta x', \Delta y', \Delta z'$ in the $x', y', z'$ using

$$\begin{aligned} \Delta x' &= -c\sin\theta\cos\alpha_{12}\sin\phi_1 + c\cos\theta\cos\phi_1 \\ \Delta y' &= c\sin\theta\sin\alpha_{12} \\ \Delta z' &= c\sin\theta\cos\alpha_{12}\cos\phi_1 + c\cos\theta\sin\phi_1 \end{aligned}$$

6. Rotate the $x',y',z'$ coordinate differences to $x,y,z$ coordinate differences by a rotation of $\lambda_1$ about the $z'$-axis using

$$\begin{aligned} \Delta x &= \Delta x'\cos\lambda_1 - \Delta y'\sin\lambda_1 \\ \Delta y &= \Delta x'\sin\lambda_1 + \Delta y'\cos\lambda_1 \\ \Delta z &= \Delta z' \end{aligned}$$

7. Compute $x,y,z$ coordinates of $P_2$ using

$$\begin{aligned} x_2 &= x_1 + \Delta x \\ y_2 &= y_1 + \Delta y \\ z_2 &= z_1 + \Delta z \end{aligned}$$

8. Compute latitude and longitude of $P_2$ by conversion $x, y, z \Rightarrow \phi, \lambda, h$ using Bowring's method.

Shown below is the output of a MATLAB function *nsection_direct.m* that solves the direct problem on the ellipsoid for normal sections.

The ellipsoid is the GRS80 ellipsoid and $\phi,\lambda$ for $P_1$ are $-10^\circ$ and $110^\circ$ respectively with $\alpha_{12} = 140^\circ\, 28'\, 31.981931''$ and $s = 5783\,228.924\,736$ m. $\phi,\lambda$ <u>computed</u> for $P_2$ are $-45^\circ$ and $155^\circ$ respectively.

```
>> nsection_direct

////////////////////////////////
// Normal Section: Direct Case //
////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257222101000
e2  =  6.694380022901e-003
ep2 =  6.694380022901e-003

Latitude  P1 =  -10  0  0.000000 (D M S)
Longitude P1 =  110  0  0.000000 (D M S)

Azimuth of normal section P1-P2
Az12  = 140 28 31.981931  (D M S)

normal section distance P1-P2
s =   5783228.924736

chord distance P1-P2
c =   5586513.169887
iterations =   13

Zenith distance of chord at P1
zd     = 116  2 20.450079  (D M S)
iterations =    5

Cartesian coordinates
            X               Y               Z
P1   -2148527.045536  5903029.542697 -1100248.547700
P2   -4094327.792179  1909216.404490 -4487348.408756
dX = -1945800.746643
dY = -3993813.138207
dZ = -3387099.861057

Latitude  P2 =  -45  0  0.000000 (D M S)
Longitude P2 =  154 59 60.000000 (D M S)

>>
```

**THE INVERSE PROBLEM ON THE ELLIPSOID USING A NORMAL SECTION**

The inverse problem is: Given latitudes and longitudes of $P_1$ and $P_2$ on the ellipsoid compute the azimuth $\alpha_{12}$ of the normal section $P_1P_2$ and the arc length $s$ of the normal section curve.

With the ellipsoid constants $a$, $f$, $e^2$ and $e'^2$ and given $\phi_1, \lambda_1$ and $\phi_2, \lambda_2$ the problem may be solved by the following sequence.

1. Compute $\nu_1$ and $\nu_2$ the radii of curvature in the prime vertical plane of $P_1$ and $P_2$ from

$$\nu = \frac{a}{\left(1 - e^2 \sin^2 \phi\right)^{\frac{1}{2}}}$$

2. Compute the $x,y,z$ coordinates of $P_1$, $P_2$, $P_3$ and $P_4$ noting that $P_3$ is at the intersection of the normal through $P_1$ and the rotational axis of the ellipsoid and $P_4$ is at the intersection of the normal through $P_2$ and the rotational axis. Coordinate of $P_1$ and $P_2$ are obtained from

$$\begin{aligned} x &= \nu \cos\phi \cos\lambda \\ y &= \nu \cos\phi \sin\lambda \\ z &= \nu\left(1 - e^2\right)\sin\phi \end{aligned}$$

The $x$ and $y$ coordinates of $P_3$ and $P_4$ are zero and the $z$ coordinate is obtained from

$$z = -\nu e^2 \sin\phi$$

3. Compute the coordinate differences

$$\begin{aligned} \Delta x &= x_2 - x_1 \\ \Delta y &= y_2 - y_1 \\ \Delta z &= z_2 - z_1 \end{aligned}$$

4a. Compute vector $\mathbf{c} = \left(\Delta x\right)\mathbf{i} + \left(\Delta y\right)\mathbf{j} + \left(\Delta z\right)\mathbf{k}$ in the direction of the chord $P_1P_2$.

4b. Compute chord distance $c = \left|\mathbf{c}\right|$ and the unit vector $\hat{\mathbf{c}} = \dfrac{\mathbf{c}}{\left|\mathbf{c}\right|}$

5. Compute vector $\mathbf{u} = \left(x_1\right)\mathbf{i} + \left(y_1\right)\mathbf{j} + \left(z_1 - z_3\right)\mathbf{k}$ and the unit vector $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{\left|\mathbf{u}\right|}$ in the direction of the outward normal through $P_1$.

6. Set the unit vector $\hat{\mathbf{z}} = 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k}$ in the direction of the $z$-axis

7. Compute the zenith distance of the chord from the vector dot product

$$\cos\theta = \hat{u}_i\hat{c}_i + \hat{u}_j\hat{c}_j + \hat{u}_k\hat{c}_k$$

8. Compute the unit vector $\hat{\mathbf{e}}$ perpendicular to the meridian plane of $P_1$ from vector cross product ($\hat{\mathbf{e}}$ is in the direction of east)

$$\hat{\mathbf{e}} = \frac{\hat{\mathbf{z}}\times\hat{\mathbf{u}}}{\cos\phi_1} = \left(\frac{\hat{z}_j\hat{u}_k - \hat{z}_k\hat{u}_j}{\cos\phi_1}\right)\mathbf{i} - \left(\frac{\hat{z}_i\hat{u}_k - \hat{z}_k\hat{u}_i}{\cos\phi_1}\right)\mathbf{j} + \left(\frac{\hat{z}_i\hat{u}_j - \hat{z}_j\hat{u}_i}{\cos\phi_1}\right)\mathbf{k}$$

9. Compute the unit vector $\hat{\mathbf{n}}$ in the meridian plane of $P_1$ from vector cross product. ($\hat{\mathbf{n}}$ is in the direction of north)

$$\hat{\mathbf{n}} = \hat{\mathbf{u}}\times\hat{\mathbf{e}} = \left(\hat{u}_j\hat{e}_k - \hat{u}_k\hat{e}_j\right)\mathbf{i} - \left(\hat{u}_i\hat{e}_k - \hat{u}_k\hat{e}_i\right)\mathbf{j} + \left(\hat{u}_i\hat{e}_j - \hat{u}_j\hat{e}_i\right)\mathbf{k}$$

10. Compute the unit vector $\hat{\mathbf{p}}$ perpendicular to the normal section $P_1P_2$ from vector cross product. ($\hat{\mathbf{p}}$ lies in the local horizon plane of $P_1$)

$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{u}}\times\hat{\mathbf{c}}}{\sin\theta} = \left(\frac{\hat{u}_j\hat{c}_k - \hat{u}_k\hat{c}_j}{\sin\theta}\right)\mathbf{i} - \left(\frac{\hat{u}_i\hat{c}_k - \hat{u}_k\hat{c}_i}{\sin\theta}\right)\mathbf{j} + \left(\frac{\hat{u}_i\hat{c}_j - \hat{u}_j\hat{c}_i}{\sin\theta}\right)\mathbf{k}$$

11. Compute the unit vector $\hat{\mathbf{g}}$ in the local horizon plane of $P_1$ and in the direction of the normal section $P_1P_2$ from vector cross product.

$$\hat{\mathbf{g}} = \hat{\mathbf{p}}\times\hat{\mathbf{u}} = \left(\hat{p}_j\hat{u}_k - \hat{p}_k\hat{u}_j\right)\mathbf{i} - \left(\hat{p}_i\hat{u}_k - \hat{p}_k\hat{u}_i\right)\mathbf{j} + \left(\hat{p}_i\hat{u}_j - \hat{p}_j\hat{u}_i\right)\mathbf{k}$$

12. Compute the azimuth $\alpha_{12}$ if the normal section $P_1P_2$ using vector dot products to first compute angles $\alpha$ (between $\hat{\mathbf{n}}$ and $\hat{\mathbf{g}}$) and $\beta$ (between $\hat{\mathbf{e}}$ and $\hat{\mathbf{g}}$) from

$$\cos\alpha = \hat{n}_i\hat{g}_i + \hat{n}_j\hat{g}_j + \hat{n}_k\hat{g}_k$$
$$\cos\beta = \hat{e}_i\hat{g}_i + \hat{e}_j\hat{g}_j + \hat{e}_k\hat{g}_k$$

If $\beta > 90^\circ$ then $\alpha_{12} = 360^\circ - \alpha$; else $\alpha_{12} = \alpha$

13. Compute the vector $\mathbf{w} = (x_1)\mathbf{i} + (y_1)\mathbf{j} + (z_1 - z_4)\mathbf{k}$ and the unit vector $\hat{\mathbf{w}} = \dfrac{\mathbf{w}}{|\mathbf{w}|}$ ($\mathbf{w}$ is in the direction of the line $P_4P_1$ and lies in the meridian plane of $P_1$).

14. Compute the angle $\gamma$ between $\hat{\mathbf{w}}$ and $\hat{\mathbf{c}}$ from the vector dot product

$$\cos\gamma = \hat{w}_i\hat{c}_i + \hat{w}_j\hat{c}_j + \hat{w}_k\hat{c}_k$$

15. Compute the angle $\delta$ between $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}}$ from the vector dot product ($\delta$ lies in the meridian plane of $P_1$)

$$\cos\delta = \hat{w}_i\hat{u}_i + \hat{w}_j\hat{u}_j + \hat{w}_k\hat{u}_k$$

16. Compute the unit vector $\hat{\mathbf{q}}$ perpendicular to the normal section $P_2P_1$ from vector cross product

$$\hat{\mathbf{q}} = \frac{\hat{\mathbf{w}}\times\hat{\mathbf{c}}}{\sin\gamma} = \left(\frac{\hat{w}_j\hat{c}_k - \hat{w}_k\hat{c}_j}{\sin\gamma}\right)\mathbf{i} - \left(\frac{\hat{w}_i\hat{c}_k - \hat{w}_k\hat{c}_i}{\sin\gamma}\right)\mathbf{j} + \left(\frac{\hat{w}_i\hat{c}_j - \hat{w}_j\hat{c}_i}{\sin\gamma}\right)\mathbf{k}$$

17. Compute the unit vector $\hat{\mathbf{h}}$ in the local horizon plane of $P_1$ and in the direction of the normal section $P_2P_1$ from vector cross product.

$$\hat{\mathbf{h}} = \frac{\hat{\mathbf{q}}\times\hat{\mathbf{u}}}{\cos\delta} = \left(\frac{\hat{q}_j\hat{u}_k - \hat{q}_k\hat{u}_j}{\cos\delta}\right)\mathbf{i} - \left(\frac{\hat{q}_i\hat{u}_k - \hat{q}_k\hat{u}_i}{\cos\delta}\right)\mathbf{j} + \left(\frac{\hat{q}_i\hat{u}_j - \hat{q}_j\hat{u}_i}{\cos\delta}\right)\mathbf{k}$$

18. Compute the azimuth $\alpha'_{12}$ of the normal section $P_2P_1$ using vector dot products to first compute angles $\alpha$ (between $\hat{\mathbf{n}}$ and $\hat{\mathbf{h}}$) and $\beta$ (between $\hat{\mathbf{e}}$ and $\hat{\mathbf{h}}$) from

$$\cos\alpha = \hat{n}_i\hat{h}_i + \hat{n}_j\hat{h}_j + \hat{n}_k\hat{h}_k$$
$$\cos\beta = \hat{e}_i\hat{h}_i + \hat{e}_j\hat{h}_j + \hat{e}_k\hat{h}_k$$

If $\beta > 90^\circ$ then $\alpha'_{12} = 360^\circ - \alpha$; else $\alpha'_{12} = \alpha$

19. Compute the small angle $\varepsilon$ between the two normal section planes at $P_1$

$$\varepsilon = \left|\alpha_{12} - \alpha'_{12}\right|$$

20. Compute arc length $s$ along the normal section curve $P_1P_2$ using Romberg Integration.

Shown below is the output of a MATLAB function *nsection_inverse.m* that solves the inverse problem on the ellipsoid for normal sections.
The ellipsoid is the GRS80 ellipsoid and $\phi,\lambda$ for $P_1$ are $-10^\circ$ and $110^\circ$ respectively and $\phi,\lambda$ for $P_2$ are $-45^\circ$ and $155^\circ$ respectively.
Computed azimuths are $\alpha_{12} = 140^\circ\, 28'\, 31.981931''$ and $\alpha'_{12} = 140^\circ\, 32'\, 18.496009''$, and $s = 5783\,228.924\,736$ m.

```
>> nsection_inverse

//////////////////////////////////
// Normal Section: Inverse Case //
//////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257222101000
e2  =  6.694380022901e-003
ep2 =  6.694380022901e-003

Latitude  P1 =  -10  0  0.000000 (D M S)
Longitude P1 =  110  0  0.000000 (D M S)

Latitude  P2 =  -45  0  0.000000 (D M S)
Longitude P2 =  155  0  0.000000 (D M S)

Cartesian coordinates
            X               Y               Z
P1   -2148527.045536  5903029.542697 -1100248.547700
P2   -4094327.792180  1909216.404490 -4487348.408755
P3          0.000000        0.000000     7415.121539
P4          0.000000        0.000000    30242.470131
dX = -1945800.746645
dY = -3993813.138206
dZ = -3387099.861055

Chord distance P1-P2
chord   =   5586513.169886

Zenith distance of chord at P1
zd      = 116  2 20.450079  (D M S)

Azimuth of normal section P1-P2
Az12    = 140 28 31.981931  (D M S)

Azimuth of normal section P2-P1
Az21    = 297 47 44.790362  (D M S)

Azimuth of normal section P2-P1 at P1
Az'12   = 140 32 18.496009  (D M S)

Angle between normal sections at P1
epsilon =    0  3 46.514078 (D M S)

ROMBERG INTEGRATION TABLE
1  5783427.529966
2  5783278.294728  5783228.549649
3  5783241.249912  5783228.901640  5783228.925106
4  5783232.004951  5783228.923298  5783228.924742  5783228.924736
5  5783229.694723  5783228.924646  5783228.924736  5783228.924736

normal section distance P1-P2
s =   5783228.924736

>>
```

## DIFFERENCE IN LENGTH BETWEEN GEODESIC AND NORMAL SECTION

There are five curves of interest in geodesy; the geodesic, the normal section, the great elliptic arc the loxodrome and the curve of alignment.

The geodesic between $P_1$ and $P_2$ on an ellipsoid is the unique curve on the surface defining the shortest distance; all other curves will be longer in length. The normal section curve $P_1P_2$ is a plane curve created by the intersection of the normal section plane containing the normal at $P_1$ and also $P_2$ with the ellipsoid surface. And as we have shown there is the other normal section curve $P_2P_1$. The curve of alignment is the locus of all points $Q$ such that the normal section plane at $Q$ also contains the points $P_1$ and $P_2$. The curve of alignment is very close to a geodesic. The great elliptic arc is the plane curve created by intersecting the plane containing $P_1$, $P_2$ and the centre $O$ with the surface of the ellipsoid and the loxodrome is the curve on the surface that cuts each meridian between $P_1$ and $P_2$ at a constant angle.

Approximate equations for the difference in length between the geodesic, the normal section curve and the curve of alignment were developed by Clarke (1880, p. 133) and Bowring (1972, p. 283) developed an approximate equation for the difference between the geodesic and the great elliptic arc. Following Bowring (1972), let

$$\begin{aligned} s &= \text{ geodesic length} \\ L &= \text{ normal section length} \\ D &= \text{ great elliptic length} \\ S &= \text{ curve of alignment length} \end{aligned}$$

then

$$\begin{aligned} L - s &= \frac{e^4}{90} s \left( \frac{s}{R} \right)^4 \cos^4 \phi_1 \sin^2 \alpha_{12} \cos^2 \alpha_{12} + \cdots \\ D - s &= \frac{e^4}{24} s \left( \frac{s}{R} \right)^2 \sin^2 \phi_1 \cos^2 \phi_1 \sin^2 \alpha_{12} + \cdots \\ S - s &= \frac{e^4}{360} s \left( \frac{s}{R} \right)^4 \cos^4 \phi_1 \sin^2 \alpha_{12} \cos^2 \alpha_{12} + \cdots \end{aligned} \tag{52}$$

where $R$ can be taken as the radius of curvature in the prime vertical at $P_1$. Now for a given value of *s*, $L - s$ will be a maximum if $\phi_1 = 0^\circ$ ($P_1$ on the equator) and $\alpha_{12} = 45^\circ$ in which case $\cos^4 \phi_1 \sin^2 \alpha_{12} \cos^2 \alpha_{12} = \dfrac{1}{4}$, thus

$$\left(L-s\right)<\frac{e^4}{360}s\left(\frac{s}{R}\right)^4 \tag{53}$$

For the GRS80 ellipsoid where $f = 1/298.257222101$, $e^2 = f\left(2-f\right)$, and for $s = 1600000$ m and $R = 6371000$ m and equation (53) gives $L - s < 0.001$ m.

This can be verified by using two MATLAB functions: *Vincenty_Direct.m* that computes the direct case on the ellipsoid for the geodesic and *nsection_inverse.m* that computes the inverse case on the ellipsoid for the normal section. Suppose $P_1$ has latitude and longitude $\phi_1 = 0^\circ$, $\lambda_1 = 0^\circ$ on the GRS80 ellipsoid and that the azimuth and distance of the geodesic are $\alpha_{12} = 45^\circ$ and $s = 1600000$ m respectively. The coordinates of $P_2$ are obtained from *Vincenty_Direct.m* as shown below. These values are then used in *nsection_direct.m* to compute the normal section azimuth and distance $P_1P_2$.

The difference $L - s = 0.000789$ m.

```
>> Vincenty_Direct

//////////////////////////////////////////////
// DIRECT CASE on ellipsoid: Vincenty's method
//////////////////////////////////////////////

ellipsoid parameters
a    =  6378137.000000000
f    = 1/298.257222101000
b    =  6356752.314140356100
e2   =  6.694380022901e-003
ep2  =  6.739496775479e-003

Latitude & Longitude of P1
latP1 =   0  0  0.000000 (D M S)
lonP1 =   0  0  0.000000 (D M S)

Azimuth & Distance P1-P2
az12 =   45  0  0.000000 (D M S)
s    =    1600000.000000

Latitude and Longitude of P2
latP2 =  10 10 33.913466 (D M S)
lonP2 =  10 16 16.528718 (D M S)

Reverse azimuth
alpha21 = 225 55  1.180693 (D M S)

>>
```

```
>> nsection_inverse

//////////////////////////////////
// Normal Section: Inverse Case //
//////////////////////////////////

ellipsoid parameters
a   =  6378137.000000000
f   = 1/298.257222101000
e2  =  6.694380022901e-003
ep2 =  6.694380022901e-003

Latitude  P1 =    0  0  0.000000 (D M S)
Longitude P1 =    0  0  0.000000 (D M S)

Latitude  P2 =   10 10 33.913466 (D M S)
Longitude P2 =   10 16 16.528718 (D M S)

Azimuth of normal section P1-P2
Az12    =  45  0  7.344646  (D M S)

ROMBERG INTEGRATION TABLE
1 1600010.313769
2 1600002.577521 1599999.998771
3 1600000.644877 1600000.000663 1600000.000789
4 1600000.161805 1600000.000781 1600000.000789
  1600000.000789

normal section distance P1-P2
s =   1600000.000789

>>
```

Differences in length between the geodesic and normal section exceed 0.001 m for distances greater than 1,600 km. At 5,800 km the difference is approximately 0.380 m.

**MATLAB FUNCTIONS**

Shown below are two MATLAB functions *nsection_ direct.m* and *nsection_ inverse.m* that have been written to demonstrate the use of Romberg integration in the solution of the direct and inverse case on the ellipsoid using normal sections. These functions call other functions; *DMS.m, Cart2Geo.m* and *romberg.m* that are also shown.

**MATLAB function *nsection_ direct.m***

```
function nsection_direct
%
% nsection_direct:  This function computes the direct case for a normal
% section on the reference ellipsoid.  That is, given the latitude and
% longitude of P1 and the azimuth of the normal section P1-P2 and distance
% along the normal section curve, compute the latitude and longitude of P2.

%--------------------------------------------------------------------------
% Function:  nsection_direct
%
% Usage:     nsection_direct
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  23 September 2009
%            Version  1.1  16 December  2009
%
% Purpose:   nsection_inverse:  This function computes the direct case for
%  a normal section on the reference ellipsoid.  That is, given the
%  latitude and longitude of P1 and the azimuth of the normal section P1-P2
%  and distance along the normal section curve, compute the latitude and
%  longitude of P2.
%
% Functions required:
%  [D,M,S] = DMS(DecDeg)
%  s = romberg(a,f,lat1,Az12,zd)
%  [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z)
%
% Variables:
%  Az12     - azimuth of normal section P1-P2
%  a        - semi-major axis of spheroid
%  d2r      - degree to radian conversion factor 57.29577951...
%  e2       - eccentricity of ellipsoid squared
%  eps      - 2nd-eccentricity squared
%  f        - f = 1/flat is the flattening of ellipsoid
%  flat     - denominator of flattening of ellipsoid
%  f_zd     - function of the zenith distance
%  fdash_zd - derivative of the function of the zenith distance
%  g,h      - constants of normal section
%  lat1     - latitude of P1 (radians)
%  lat2     - latitude of P2 (radians)
%  lon1     - longitude of P1 (radians)
%  lon2     - longitude of P2 (radians)
%  nu1      - radius of curvature in prime vertical plane at P1
%  pion2    - pi/2
%  s        - arc length of normal section P1-P2
%  s2       - sin-squared(latitude)
%  x,y      - local variables in newton-Raphson iteration for zenith
%             distance of chord P1-P2
%  X1,Y1,Z1 - Cartesian coordinates of P1
```

```
%  X2,Y2,Z2 - Cartesian coordinates of P2
%  X3,Y3,Z3 - Cartesian coordinates of P3
%  X4,Y4,Z4 - Cartesian coordinates of P4
%  zd       - zenith distance of chord
%
% Remarks:
%
% References:
%  [1] Deakin, R. E., (2009), "The Normal Section Curve on an Ellipsoid",
%          Lecture Notes, School of Mathematical and Geospatial Sciences,
%          RMIT University, November 2009.
%
%--------------------------------------------------------------------------

% Set degree to radian conversion factor and pi/2
d2r   = 180/pi;
pion2 = pi/2;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Compute ellipsoid constants
f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);

% Set lat and long of P1 on ellipsoid
lat1 = -10/d2r;
lon1 = 110/d2r;

% Set azimuth of normal section P1-P2 and arc length of normal section
Az12 = (140 + 28/60 + 31.981931/3600)/d2r;
s    = 5783228.924736;

% [1] Compute radius of curvature in the prime vertical plane at P1
s2   = sin(lat1)^2;
nu1  = a/sqrt(1-e2*s2);

% [2] Compute constants g and h of the normal section P1-P2
ep = sqrt(ep2);
g  = ep*sin(lat1);
h  = ep*cos(lat1)*cos(Az12);

% [3] Compute the chord and the zenith distance of the chord of the normal
%     section curve P1-P2 by iteration.

% Set the chord equal to the arc length
c = s;
iter_1 = 1;
while 1
    % Set the zenith distance to 90 degrees
    zd = pion2;
    % Compute the zenith distance of the chord using Newton-Raphson iteration
    iter_2 = 1;
    while 1
        x = g*cos(zd)+h*sin(zd);
        y = h*cos(zd)-g*sin(zd);
        f_zd = c+c*x*x+2*nu1*cos(zd);
        fdash_zd = 2*c*x*y-2*nu1*sin(zd);
        new_zd = zd-(f_zd/fdash_zd);
        if abs(new_zd - zd) < 1e-15
            break;
        end
        zd = new_zd;
        if iter_2 > 10
            fprintf('Iteration for zenith distance failed to converge after 10
iterations');
            break;
```

```
        end
        iter_2 = iter_2 + 1;
    end;
    % Compute normal section arc length for zenith distance
    s_new = romberg(a,f,lat1,Az12,zd);
    ds = s_new-s;
    if abs(ds) < 1e-6
        break;
    end
    c = c - ds;
    if iter_1 > 15
        fprintf('Iteration for chord distance failed to converge after 15 iterations');
        break;
    end
    iter_1 = iter_1 + 1;
end;

% [4] Compute X,Y,Z Cartesian coordinates of P1
X1 = nu1*cos(lat1)*cos(lon1);
Y1 = nu1*cos(lat1)*sin(lon1);
Z1 = nu1*(1-e2)*sin(lat1);

% [5] Compute X',Y',Z' coord differences with Z'-X' plane coincident with meridian
%     plane of P1
dXp = -c*sin(zd)*cos(Az12)*sin(lat1) + c*cos(zd)*cos(lat1);
dYp =  c*sin(zd)*sin(Az12);
dZp =  c*sin(zd)*cos(Az12)*cos(lat1) + c*cos(zd)*sin(lat1);

% [6] Rotate X',Y',Z' coord differences by lon1 about Z'-axis
dX = dXp*cos(lon1) - dYp*sin(lon1);
dY = dXp*sin(lon1) + dYp*cos(lon1);
dZ = dZp;

% [7] Compute X,Y,Z coords of P2
X2 = X1 + dX;
Y2 = Y1 + dY;
Z2 = Z1 + dZ;

% [8] Compute lat, lon and ellipsoidal height of P2 using Bowring's method
[lat2,lon2,h2] = Cart2Geo(a,flat,X2,Y2,Z2);

%-----------------------
% Print result to screen
%-----------------------

fprintf('\n//////////////////////////////////');
fprintf('\n// Normal Section: Direct Case //');
fprintf('\n//////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na   = %18.9f',a);
fprintf('\nf   = 1/%16.12f',flat);
fprintf('\ne2  = %20.12e',e2);
fprintf('\nep2 = %20.12e',e2);

% Print lat and lon of P1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print azimuth of normal section
```

```
fprintf('\n\nAzimuth of normal section P1-P2');
[D,M,S] = DMS(Az12*d2r);
fprintf('\nAz12  = %3d %2d %9.6f  (D M S)',D,M,S);

% Print normal section distance P1-P2
fprintf('\n\nnormal section distance P1-P2');
fprintf('\ns =  %15.6f',s);

% Print chord distance P1-P2
fprintf('\n\nchord distance P1-P2');
fprintf('\nc =  %15.6f',c);
fprintf('\niterations = %4d',iter_1);

% Print zenith distance of chord at point 1
fprintf('\n\nZenith distance of chord at P1');
[D,M,S] = DMS(zd*d2r);
fprintf('\nzd     = %3d %2d %9.6f  (D M S)',D,M,S);
fprintf('\niterations = %4d',iter_2);


% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n            X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);
fprintf('\ndX = %15.6f',dX);
fprintf('\ndY = %15.6f',dY);
fprintf('\ndZ = %15.6f',dZ);

% Print lat and lon of P2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat2 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

fprintf('\n\n');
```

## MATLAB function *nsection_ inverse.m*

```
function nsection_inverse
%
% nsection_inverse:  This function computes the inverse case for a normal
% section on the reference ellipsoid.  That is, given the latitudes and
% longitudes of two points on the ellipsoid, compute the azimuth and the
% arc length of the normal section.

%---------------------------------------------------------------------------
% Function:  nsection_inverse()
%
% Usage:     nsection_inverse
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  21 September 2009
%            Version  1.1  16 December  2009
%
% Purpose:   nsection_inverse:  This function computes the inverse case for
%  a normal section on the reference ellipsoid.  That is, given the
%  latitudes and longitudes of two points on the ellipsoid, compute the
%  azimuth and the arc length of the normal section.
%
% Functions required:
%  [D,M,S] = DMS(DecDeg)
%
% Variables:
%  alpha       - angle in the local horizon plane measured from north
%  Az12        - azimuth of normal section P1-P2
%  Azdash12    - azimuth of normal section plane P2-P1 measured at P1
%  Az21        - azimuth of normal section P2-P1
%  a           - semi-major axis of spheroid
%  beta        - angle in the local horizon plane measured from east
%  chord       - chord distance between P1 and P2
%  ci,cj,ck    - components of unit vector c in the direction of the chord
%                P1-P2
%  delta       - angle in the meridian plane of P1 between w and u vectors
%  diff        - difference between successive value of integral in Romber
%                Integration
%  du,dv,dr    - derivatives in Romberg Integration
%  dX,dY,dZ    - Cartesian components of chord between between P1 and P2
%  d2r         - degree to radian conversion factor 57.29577951...
%  ei,ej,ek    - components of unit vector e in the direction of east in
%                local horizon system
%  epsilon     - small angle between azimuths of normal section planes
%  ep2         - 2nd-eccentricity squared
%  e2          - 1st-eccentricity squared
%  f           - f = 1/flat is the flattening of ellipsoid
%  finish      - integer flag (1 or 0) to test for end of Romberg
%                Integration
%  first       - first value in trapezoidal rule in Romberg Integration
%  flat        - denominator of flattening of ellipsoid
%  gamma       - angle between unit vectors w and c
%  g,h         - constants of normal section curve
%  hi,hj,hk    - components of unit vector h in the local horizon plane and
%                direction of the plane P1-P2-P4
%  Integral    - value of integral from trapezoidal rule in Romberg
%                Integration
%  inc         - interval width in trapezoidal rule
%  int         - number of intervals in trapezoidal rule where int = 2^k
%                and k = 1:m
%  j,k         - integer counters in Romberg Integration
%  last        - last value in trapezoidal rule in Romberg Integration
%  lat1        - latitude of P1 (radians)
%  lat2        - latitude of P2 (radians)
```

```
%  lon1        - longitude of P1 (radians)
%  lon2        - longitude of P2 (radians)
%  m           - maximum power of 2 to determine number of intervals in
%                trapezoidal rule
%  norm        - length of vector
%  nu1, nu2    - radii of curvature in prime vertical plane at P1 and P2
%  ni,nj,nk    - components of unit vector n
%  pion2       - pi/2
%  qi,qj,qk - components of unit vector q perpendicular to plane
%                P1-P2-P4
%  r           - polar coordinate in polar equation of normal section
%  S           - n,n array of Integrals in Romberg Integration
%  sum         - summation in trapezoidal rule
%  s2          - sin-squared(latitude)
%  ui,uj,uk    - components of unit vector u
%  wi,wj,wk    - components of unit vector w
%  x,y         - variables in Romberg Integration
%  X1,Y1,Z1    - Cartesian coordinates of P1
%  X2,Y2,Z2    - Cartesian coordinates of P2
%  X3,Y3,Z3    - Cartesian coordinates of P3
%  X4,Y4,Z4    - Cartesian coordinates of P4
%  zd          - zenith distance of chord
%
% Remarks:
%  P1 and P2 are two point on the ellipsoid and in general there are two
%  normal section curves between them.  P3 is at the intersection of the
%  rotational axis of the ellipsoid and the normal through P1.  P4 is at
%  the intersection of the rotational axis of the ellipsoid and the normal
%  through P2.  The normal section P1-P2 is the plane P1-P2-P3.  The normal
%  section P2-P1 is the plane P1-P2-P4 and since P3 and P4 are not
%  coincident (in general) then the two planes create two lines on the
%  ellipsoid and two lines on the local horizon plane at P1.
%  The necessary equations for the solution of the inverse problem (normal
%  sections) on the ellipsoid are described in [1].  The vector
%  manipulations to determine the difference between the two normal section
%  plane azimuths (measuered in the local horizon at P1) follows a vector
%  method of calculating azimuth given in [2].
%  This function uses Romberg Integration to compute the arc length along
%  the normal section curve.  This technique of numerical integration is
%  described in detail in [1].
%
% References:
%  [1] Deakin, R. E., (2009), "The Normal Section Curve on an Ellipsoid",
%          Lecture Notes, School of Mathematical and Geospatial Sciences,
%          RMIT University, November 2009.
%  [2] Deakin, R. E., (1988), "The Determination of the Instantaneous
%          Position of the NIMBUS-7 CZCS Satellite", Symposium on Remote
%          Sensing of the Coastal Zone, Queensland, 1988.
%
%-------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;
pion2 = pi/2;

% Set ellipsoid parameters
a    = 6378137;    % GRS80
flat = 298.257222101;

% Compute ellipsoid constants
 f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);

% Set lat and long of P1 and P2 on ellipsoid
lat1 = -10/d2r;
lon1 =  110/d2r;
lat2 = -45/d2r;
lon2 =  155/d2r;
```

```
% [1] Compute radii of curvature in the prime vertical plane at P1 & P2
s2  = sin(lat1)^2;
nu1 = a/sqrt(1-e2*s2);
s2  = sin(lat2)^2;
nu2 = a/sqrt(1-e2*s2);

% [2] Compute Cartesian coordinates of points P1, P2, P3 and P4
% Note that P3 is at the intesection of the normal through P1 and
% the rotational axis and P4 is at the intersection of the normal
% through P2 and the rotational axis.
X1 = nu1*cos(lat1)*cos(lon1);
Y1 = nu1*cos(lat1)*sin(lon1);
Z1 = nu1*(1-e2)*sin(lat1);

X2 = nu2*cos(lat2)*cos(lon2);
Y2 = nu2*cos(lat2)*sin(lon2);
Z2 = nu2*(1-e2)*sin(lat2);

X3 = 0;
Y3 = 0;
Z3 = -nu1*e2*sin(lat1);

X4 = 0;
Y4 = 0;
Z4 = -nu2*e2*sin(lat2);

% [3] Compute coordinate differences that are the components of the chord
% P1-P2
dX = X2 - X1;
dY = Y2 - Y1;
dZ = Z2 - Z1;

% [4a] Compute the vector c in the direction of the chord between P1 and P2
ci = dX;
cj = dY;
ck = dZ;

% [4b] Compute the chord distance and the unit vector c
chord = sqrt(ci*ci + cj*cj + ck*ck);
ci = ci/chord;
cj = cj/chord;
ck = ck/chord;

% [5] Compute the unit vector u in the direction of the normal through P1
ui = X1;
uj = Y1;
uk = Z1-Z3;
norm = sqrt(ui*ui + uj*uj + uk*uk);
ui = ui/norm;
uj = uj/norm;
uk = uk/norm;

% [6] Set unit vector for the z-axis of ellipsoid
zi = 0;
zj = 0;
zk = 1;

% [7] Compute zenith distance of chord at P1 from dot product
zd = acos(ui*ci + uj*cj + uk*ck);

% [8] Compute unit vector e perpendicular to meridian plane using vector cross
% product e = (z x u)/cos(lat1).  e is in the direction of east.
ei =  (zj*uk - zk*uj)/cos(lat1);
ej = -(zi*uk - zk*ui)/cos(lat1);
ek =  (zi*uj - zj*ui)/cos(lat1);

% [9] Compute unit vector n in the meridian plane using vector cross
% product n = u x e.  n is in the direction of north.
```

```
ni =  (uj*ek - uk*ej);
nj = -(ui*ek - uk*ei);
nk =  (ui*ej - uj*ei);

% [10] Compute unit vector p perpendicular to normal section P1-P2 using
% vector cross product q = (u x c)/sin(zd)
pii =  (uj*ck - uk*cj)/sin(zd);
pj = -(ui*ck - uk*ci)/sin(zd);
pk =  (ui*cj - uj*ci)/sin(zd);

% [11] Compute unit vector g in the local horizon plane of P1 and in the
% direction of the normal section P1-P2 using vector cross product
% g = p x u
gi =  (pj*uk - pk*uj);
gj = -(pii*uk - pk*ui);
gk =  (pii*uj - pj*ui);

% [12] Compute azimuth of normal section P1-P2-P3 using vector dot product
alpha = acos(ni*gi + nj*gj + nk*gk);
beta  = acos(ei*gi + ej*gj + ek*gk);

if beta > pi/2
    Az12 = 2*pi - alpha;
else
    Az12 = alpha;
end

% [13] Compute unit vector w in direction of line P4-P1.  w will lie in the
% meridian plane of P1.
wi = X1;
wj = Y1;
wk = Z1-Z4;
norm = sqrt(wi*wi + wj*wj + wk*wk);
wi = wi/norm;
wj = wj/norm;
wk = wk/norm;

% [14] Compute the angle gamma between unit vectors w and c using vector
% dot product gamma = acos(w . c)
gamma = acos(wi*ci + wj*cj + wk*ck);

% [15] Compute the angle delta between unit vectors w and u using vector
% dot product delta = acos(w . u)
delta = acos(wi*ui + wj*uj + wk*uk);

% [16] Compute unit vector q perpendicular to plane P2-P1-P4 using vector
% cross product q = (w x c)/sin(gamma)
qi =  (wj*ck - wk*cj)/sin(gamma);
qj = -(wi*ck - wk*ci)/sin(gamma);
qk =  (wi*cj - wj*ci)/sin(gamma);

% [17] Compute unit vector h in the direction of P2 and in the local horizon
% plane using vector cross product h = (q x u)/cos(delta)
hi =  (qj*uk - qk*uj)/cos(delta);
hj = -(qi*uk - qk*ui)/cos(delta);
hk =  (qi*uj - qj*ui)/cos(delta);

% [18] Compute azimuth of section P1-P2-P4 using vector dot product
alpha = acos(ni*hi + nj*hj + nk*hk);
beta  = acos(ei*hi + ej*hj + ek*hk);

if beta > pi/2
    Azdash12 = 2*pi - alpha;
else
    Azdash12 = alpha;
end

% [19] Compute angle between normal section planes at P1
epsilon = abs(Az12-Azdash12);
```

```
% Compute normal section azimuth P2 to P1
numerator   = dX*sin(lon2) - dY*cos(lon2);
denominator = dX*sin(lat2)*cos(lon2)  + dY*sin(lat2)*sin(lon2) - dZ*cos(lat2);
Az21 = atan2(numerator,denominator);
if Az21 < 0
    Az21 = 2*pi+Az21;
end

%-----------------------
% Print result to screen
%-----------------------
fprintf('\n////////////////////////////////////');
fprintf('\n// Normal Section: Inverse Case //');
fprintf('\n////////////////////////////////////');
fprintf('\n\nellipsoid parameters');
fprintf('\na   = %18.9f',a);
fprintf('\nf   = 1/%16.12f',flat);
fprintf('\ne2  = %20.12e',e2);
fprintf('\nep2 = %20.12e',e2);

% Print lat and lon of Point 1
[D,M,S] = DMS(lat1*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P1 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon1*d2r);
if D == 0 && lon1 < 0
    fprintf('\nLongitude P1 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P1 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print lat and lon of point 2
[D,M,S] = DMS(lat2*d2r);
if D == 0 && lat1 < 0
    fprintf('\n\nLatitude  P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\n\nLatitude  P2 = %4d %2d %9.6f (D M S)',D,M,S);
end
[D,M,S] = DMS(lon2*d2r);
if D == 0 && lon2 < 0
    fprintf('\nLongitude P2 =   -0 %2d %9.6f (D M S)',M,S);
else
    fprintf('\nLongitude P2 = %4d %2d %9.6f (D M S)',D,M,S);
end

% Print Coordinate table
fprintf('\n\nCartesian coordinates');
fprintf('\n            X               Y               Z');
fprintf('\nP1   %15.6f %15.6f %15.6f',X1,Y1,Z1);
fprintf('\nP2   %15.6f %15.6f %15.6f',X2,Y2,Z2);
fprintf('\nP3   %15.6f %15.6f %15.6f',X3,Y3,Z3);
fprintf('\nP4   %15.6f %15.6f %15.6f',X4,Y4,Z4);
fprintf('\ndX = %15.6f',dX);
fprintf('\ndY = %15.6f',dY);
fprintf('\ndZ = %15.6f',dZ);

% Print chord distance 1-2
fprintf('\n\nChord distance P1-P2');
fprintf('\nchord   =  %15.6f',chord);

% Print zenith distance of chord at point 1
fprintf('\n\nZenith distance of chord at P1');
[D,M,S] = DMS(zd*d2r);
fprintf('\nzd      = %3d %2d %9.6f  (D M S)',D,M,S);
```

```
% Print azimuths of normal sections
fprintf('\n\nAzimuth of normal section P1-P2');
[D,M,S] = DMS(Az12*d2r);
fprintf('\nAz12    = %3d %2d %9.6f  (D M S)',D,M,S);

fprintf('\n\nAzimuth of normal section P2-P1');
[D,M,S] = DMS(Az21*d2r);
fprintf('\nAz21    = %3d %2d %9.6f  (D M S)',D,M,S);

fprintf('\n\nAzimuth of normal section P2-P1 at P1');
[D,M,S] = DMS(Azdash12*d2r);
fprintf('\nAz''12   = %3d %2d %9.6f  (D M S)',D,M,S);

fprintf('\n\nAngle between normal sections at P1');
[D,M,S] = DMS(epsilon*d2r);
fprintf('\nepsilon = %4d %2d %9.6f (D M S)',D,M,S);

% [20] Compute arc length of normal section using ROMBERG INTEGRATION

fprintf('\n\nROMBERG INTEGRATION TABLE');

% Compute constants of normal section curve P1-P2
ep = sqrt(ep2);
g  = ep*sin(lat1);
h  = ep*cos(lat1)*cos(Az12);
m = 15;
S = zeros(m,m);
finish = 0;
for k = 1:m
    int = 2^k;
    inc = (zd-pion2)/int;
    sum = 0;
    for t = pion2:inc:zd
        x = g*cos(t)+h*sin(t);
        y = h*cos(t)-g*sin(t);
        u = -2*nu1*cos(t);
        v = 1+x*x;
        r = u/v;
        du = 2*nu1*sin(t);
        dv = 2*x*y;
        dr = (v*du-u*dv)/(v*v);
        y = sqrt(r*r + dr*dr);
        sum = sum+2*y;
        if t == pion2
            first = y;
        end
        last = y;
    end
    sum = sum-first-last;
    Integral = inc/2*sum;
    S(k,1) = Integral;
    fprintf('\n%d %15.6f',k,S(k,1));
    for j = 2:k
        S(k,j) = 1/(4^(j-1)-1)*(4^(j-1)*S(k,j-1)-S(k-1,j-1));
        fprintf(' %15.6f',S(k,j));
        diff = abs(S(k,j-1)-S(k,j));
        if diff < 1e-6
            finish = 1;
            s = S(k,j);
            break;
        end
    end
    if finish == 1
        break;
    end
end

% Print normal section distance P1-P2
fprintf('\n\nnormal section distance P1-P2');
```

```
fprintf('\ns =  %15.6f',s);

fprintf('\n\n');
```

## MATLAB function *Cart2Geo.m*

```
function [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z)
%
% [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z)
%   Function computes the latitude (lat), longitude (lon) and height (h)
%   of a point related to an ellipsoid defined by semi-major axis (a)
%   and denominator of flattening (flat) given Cartesian coordinates
%   X,Y,Z.  Latitude and longitude are returned as radians.

%--------------------------------------------------------------------------
% Function:  Cart2Geo()
%
% Usage:     [lat,lon,h] = Cart2Geo(a,flat,X,Y,Z);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0   6 April  2006
%            Version  1.1  20 August 2007
%
% Functions required:
%    radii()
%
% Purpose:
%    Function Cart2geo() will compute latitude, longitude
%    (both in radians) and height of a  point related to
%    an ellipsoid defined by semi-major axis (a) and
%    denominator of flattening (flat) given Cartesian coordinates
%    X,Y,Z.
%
% Variables:
%    a       - semi-major axis of ellipsoid
%    b       - semi-minor axis of ellipsoid
%    c       - cos(psi)
%    c3      - cos(psi) cubed
%    e2      - 1st eccentricity squared
%    ep2     - 2nd eccentricity squared
%    f       - flattening of ellipsoid
%    flat    - denominator of flattening f = 1/flat
%    h       - height above ellipsoid
%    lat     - latitude (radians)
%    lon     - longitude (radians)
%    p       - perpendicular distance from minor-axis of ellipsoid
%    psi     - parametric latitude (radians)
%    rm      - radius of curvature of meridian section of ellipsoid
%    rp      - radius of curvature of prime vertical section of ellipsoid
%    s       - sin(psi)
%    s3      - sin(psi) cubed
%
% Remarks:
%    This function uses Bowring's method, see Ref [1].
%    Bowring's method is also explained in Ref [2].
%
% References:
% [1] Bowring, B.R., 1976, 'Transformation from spatial to
%     geographical coordinates', Survey Review, Vol. XXIII,
%     No. 181, pp. 323-327.
% [2] Gerdan, G.P. & Deakin, R.E., 1999, 'Transforming Cartesian
%     coordinates X,Y,Z to geogrpahical coordinates phi,lambda,h', The
%     Australian Surveyor, Vol. 44, No. 1, pp. 55-63, June 1999.
%--------------------------------------------------------------------------
```

```
% calculate flattening f and ellipsoid constants e2, ep2 and b
f   = 1/flat;
e2  = f*(2-f);
ep2 = e2/(1-e2);
b   = a*(1-f);

% compute 1st approximation of parametric latitude psi
p   = sqrt(X*X + Y*Y);
psi = atan((Z/p)/(1-f));

% compute latitude from Bowring's equation
s   = sin(psi);
s3  = s*s*s;
c   = cos(psi);
c3  = c*c*c;
lat = atan((Z+b*ep2*s3)/(p-a*e2*c3));

% compute radii of curvature for the latitude
[rm,rp] = radii(a,flat,lat);

% compute longitude and height
lon = atan2(Y,X);
h   = p/cos(lat) - rp;


function [D,M,S] = DMS(DecDeg)
% [D,M,S] = DMS(DecDeg)  This function takes an angle in decimal degrees and returns
%   Degrees, Minutes and Seconds

val = abs(DecDeg);
D = fix(val);
M = fix((val-D)*60);
S = (val-D-M/60)*3600;
if(DecDeg<0)
  D = -D;
end
return
```

### MATLAB function *romberg.m*

```
function s = romberg(a,f,lat1,Az12,zd)
%
% s = romberg(a,f,lat,az,zd)
% This function cumputes the arc length of a normal section using Romberg
% Integration, a numerical integration technique using the trapezoidal rule
% and Richardson Extrapolation.  The function requires ellipsoid parameters
% a (semi-major axis) and f (flattening of ellipsoid), lat1 (latitude of P1
% in radians), Az12 (azimuth of normal section plane P1-P2 in radians) and
% zd (zenith distance of the chord of the normal section arc P1-P2).  The
% function returns the arc length s.

%--------------------------------------------------------------------------
% Function:  romberg
%
% Usage:     s = romberg(a,f,lat1,Az12,zd);
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  24 September 2009
%
% Purpose:   This function cumputes the arc length of a normal section
%  using Romberg Integration, a numerical integration technique using the
%  trapezoidal rule and Richardson Extrapolation.  The function requires
```

```
%  ellipsoid parameters a,f and lat1 (latitude of P1 in radians), Az12
%  (azimuth of normal section plane P1-P2 in radians) and zd (zenith
%  distance of the chord of the normal section arc P1-P2).
%
% Functions required:
%
% Variables:
%  Az12     - azimuth of normal section P1-P2
%  a        - semi-major axis of spheroid
%  chord    - chord distance between P1 and P2
%  d2r      - degree to radian conversion factor 57.29577951...
%  e2       - eccentricity of ellipsoid squared
%  eps      - 2nd-eccentricity squared
%  f        - f = 1/flat is the flattening of ellipsoid
%  g,h      - constants of normal section curve
%  lat1     - latitude of P1 (radians)
%  nu1      - radius of curvature in prime vertical plane at P1
%  pion2    - pi/2
%  S        - array of normal section arc lengths
%  s        - arc length of normal section P1-P2
%  s2       - sin-squared(latitude)
%  zd       - zenith distance of chord
%
% Remarks:
%
% References:
%  [1] Deakin, R. E., (2009), "The Normal Section Curve on an Ellipsoid",
%          Lecture Notes, School of Mathematical and Geospatial Sciences,
%          RMIT University, November 2009.
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;
pion2 = pi/2;

% Compute ellipsoid constants
e2  = f*(2-f);
ep2 = e2/(1-e2);

% Compute radius of curvature in the prime vertical plane at P1
s2  = sin(lat1)^2;
nu1 = a/sqrt(1-e2*s2);

%---------------------------------------------------------------
% Compute arc length of normal section using ROMBERG INTEGRATION
%---------------------------------------------------------------

% fprintf('\n\nROMBERG INTEGRATION TABLE');

% Compute constants of normal section curve P1-P2
ep = sqrt(ep2);
g  = ep*sin(lat1);
h  = ep*cos(lat1)*cos(Az12);

% Set array of arc lengths
n = 15;
S = zeros(n,n);
finish = 0;
for k = 1:15
    % set the number of intervals and the increment
    int = 2^k;
    inc = (zd-pion2)/int;
    sum = 0;
    % evaluate the integral using the Trapezoidal Rule
    for t = pion2:inc:zd
        x = g*cos(t)+h*sin(t);
        y = h*cos(t)-g*sin(t);
        u = -2*nu1*cos(t);
```

```
        v = 1+x*x;
        r = u/v;
        du = 2*nu1*sin(t);
        dv = 2*x*y;
        dr = (v*du-u*dv)/(v*v);
        y = sqrt(r*r + dr*dr);
        sum = sum+2*y;
        if t == pion2
            first = y;
        end
        last = y;
    end
    sum = sum-first-last;
    Integral = inc/2*sum;
    S(k,1) = Integral;
%    fprintf('\n%d %15.6f',k,S(k,1));
    % Use Richardson extrapolation
    for j = 2:k
        S(k,j) = 1/(4^(j-1)-1)*(4^(j-1)*S(k,j-1)-S(k-1,j-1));
%        fprintf(' %15.6f',S(k,j));
        diff = abs(S(k,j-1)-S(k,j));
        if diff < 1e-6
            finish = 1;
            s = S(k,j);
            break;
        end
    end
    if finish == 1
        break;
    end
end
```

**REFERENCES**


Bowring, B. R., (1972), 'Distance and the spheroid', Correspondence, *Survey Review,* Vol. XXI, No. 164, April 1972, pp. 281-284.

Bowring, B. R., (1978), 'The surface controlled spatial system for surveying computations', *Survey Review,* Vol. XXIIII, No. 190, October 1978, pp. 361-372.

Clarke, A. R., (1880), *Geodesy,* Clarendon Press, Oxford.

Deakin, R. E. and Hunter, M. N., (2007), 'Geodesics on an ellipsoid – Pittman's method', *Proceedings of the Spatial Sciences Institute Biennial International Conference (SSC2007),* Hobart, Tasmania, Australia, 14-18 May 2007, pp. 223-242.

Deakin, R. E. and Hunter, M. N., (2007), 'Geodesics on an ellipsoid – Bessel's Method', Lecture Notes, School of Mathematical & Geospatial Sciences, RMIT University, Melbourne, Australia, 66 pages.

Dutka, J., (1984), 'Richardson extrapolation and Romberg integration', *Historia Mathematica,* Vol. 11, Issue 1, February 1984, pp. 3-21. doi: 10.1016/0315-0860(84)90002-8.

Grossman, S. I., (1981), *Calculus,* 2nd edition, Academic Press, New York.

Romberg, W., (1955), 'Vereinfachte numerische integration', *Det Kongelige Norske Videnskabers Selskab Forhandlinger* (Trondheim), Vol. 28, No. 7, pp. 30-36.

Tobey, W. M., (1928), *Geodesy,* Geodetic Survey of Canada Publications No. 11, Ottawa 1928.

Williams, P. W., (1972), *Numerical Computation,* Nelson, London.

## APPENDIX 1: ROMBERG INTEGRATION

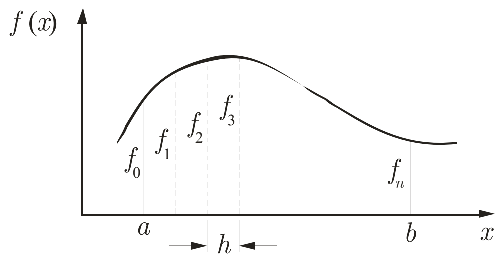
Romberg integration (Romberg 1955) is a numerical technique for evaluating a definite integral and discussions of the technique can be found in most textbooks on numerical analysis; e.g. Williams (1972). A concise treatment of the technique and a study of the historical development of methods of integration (quadrature) can be found in Dutka (1984). A development of Romberg's method – and the extrapolation formula that is at the heart of it – is given below and is followed by a MATLAB function that demonstrates the use of the technique.

Romberg integration is a method for estimating the numerical value of the definite integral

$$I = \int_a^b f\left(x\right)dx \tag{54}$$

It is based on the trapezoidal rule – the simplest of the Newton-Cotes integration formula for equally spaced data on the interval $a,b$

$$I = \int_a^b f\left(x\right)dx = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right) + E \tag{55}$$




where

$n$ is the number of intervals of width $h$,

$h = \dfrac{b-a}{n}$ is the common interval width or spacing,

$f_0, f_1, f_2, \ldots$ are values of the function evaluated at $x = \left[a,\ a+h,\ a+2h,\ \ldots\right]$,

$E$ is the error term

When the function $f\left(x\right)$ has continuous derivatives the error term $E$ can be expressed as a convergent power series and we may write

$$I = \int_a^b f\left(x\right)dx = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right) + E = T + \sum_{j=1}^{\infty} a_j h^{2j} \tag{56}$$

where $a_j$ are coefficients.

As the error term $E$ is a convergent power series in $h$ a technique known as Richardson extrapolation[1] may be employed to improve the accuracy of the result.

Richardson extrapolation can be explained as follows.

Let the value of $n$ be a power of 2; say $2^k$ i.e., the number of intervals $n = 2, 4, 8, 16, \ldots, 2^k$

Denote an evaluation of the integral $I$ given by equation (56) as

$$S_{k,1} = T + \sum_{j=1}^{\infty} a_j h^{2j} = T + a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots \tag{57}$$

If the interval width is halved, then

$$S_{k+1,1} = T + \sum_{j=1}^{\infty} a_j \left(\frac{h}{2}\right)^{2j} = T + a_1 \frac{1}{2^2} h^2 + a_2 \frac{1}{2^4} h^4 + a_3 \frac{1}{2^6} h^6 + \cdots \tag{58}$$

The first term of the error series can be eliminated by taking suitable combinations of equations (57) and (58); i.e., multiplying equation (58) by 4 and then subtracting equation (57) will eliminate the first term of the error series

$$\begin{aligned} 4S_{k+1,1} - S_{k,1} &= 4T - T + a_2 \left(\frac{4h^4}{2^4} - h^4\right) + a_3 \left(\frac{4h^6}{2^6} - h^6\right) + \cdots \\ &= 3T + \sum_{j=2}^{\infty} a_j \left(\frac{4h^{2j}}{2^{2j}} - h^{2j}\right) \end{aligned}$$

and

$$T = \frac{4S_{k+1,1} - S_{k,1}}{3} - \sum_{j=2}^{\infty} \frac{a_j h^{2j}}{3} \left(\frac{4}{2^{2j}} - 1\right) \tag{59}$$

The first term on the right-hand-side of equation (59) will be designated

$$S_{k,2} = \frac{4S_{k+1,1} - S_{k,1}}{3}$$

and the leading error term is now of order $h^4$.

[1] A technique named after Lewis Fry Richardson (1881–1953) a British applied mathematician, physicist, meteorologist, psychologist and pacifist who developed the numerical methods used in weather forecasting and also applied his mathematical techniques to the analysis of the causes and prevention of wars. He was also a pioneer in the study of fractals. Richardson extrapolation is also known as Richardson's deferred approach to the limit.

Successive halvings of the interval will give a sequence of values $S_{1,1}, S_{2,1}, S_{3,1}, \ldots, S_{k,1}$ and each successive pair $\left(S_{1,1}, S_{2,1}\right), \left(S_{2,1}, S_{3,1}\right), \ldots$ can be combined to give values $S_{2,2}, S_{3,2}, \ldots$; and this next sequence can be combined in a similar manner to remove the leading error term of order $h^4$; and so on.

By using the formula

$$S_{k,j} = \frac{1}{4^{j-1} - 1}\left(4^{j-1} S_{k,j-1} - S_{k-1,j-1}\right) \qquad \begin{array}{l} k = 1, 2, 3, 4, \ldots \\ j = 2, 3, 4, 5, \ldots \end{array} \tag{60}$$

the process of Richardson extrapolation leads to a triangular sequence of columns with error terms of increasing order.

| $n$ | $k$ \ $j$ | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|
| 2 | 1 | $S_{1,1}$ | | | | |
| 4 | 2 | $S_{2,1}$ | $S_{2,2}$ | | | |
| 16 | 3 | $S_{3,1}$ | $S_{3,2}$ | $S_{3,3}$ | | |
| 32 | 4 | $S_{4,1}$ | $S_{4,2}$ | $S_{4,3}$ | $S_{4,4}$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| error term | | $h^2$ | $h^4$ | $h^6$ | $h^8$ | |

The entries $S_{k,2}$ in the second column have eliminated the terms involving $h^2$, the entries in the third column have eliminated the terms involving $h^4$, etc, and as the interval $h = \dfrac{b-a}{2^k}$ the error term of the approximation $S_{k,j}$ is of the order $\left(\dfrac{b-a}{2^k}\right)^{2j}$ with each successive value in a particular row converging more rapidly to the true value of the integral.

Testing between particular values will determine when the process has converged to a suitable result.

**MATLAB FUNCTION** *romberg_ test.m*

This function uses Romberg Integration for the calculation of the integral $\int \sec\left(x\right)dx$

This integral has the known result $\int \sec\left(x\right)dx = \ln\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$

MATLAB function *romberg_ test.m*

```
function romberg_test
%
% This function computes the numerical value of the integral of sec(x)
% which is known to equal ln[tan(x/2+pi/4)].
% For x = 45 degrees the integral sec(x) = 0.881373587020.
% An integration table is produced that shows the convergence to the true
% value of the integral.

%--------------------------------------------------------------------------
% Function:  romberg_test
%
% Author:    R.E.Deakin,
%            School of Mathematical & Geospatial Sciences, RMIT University
%            GPO Box 2476V, MELBOURNE, VIC 3001, AUSTRALIA.
%            email: rod.deakin@rmit.edu.au
%            Version  1.0  09 December 2009
%
% Purpose:   This function computes the numerical value of the integral of
%  sec(x) which is known to equal ln[tan(x/2+pi/4)].
%  For x = 45 degrees the integral sec(x) = 0.881373587.
%  An integration table is produced that shows the convergence to the true
%  value of the integral.
%
% Variables:
%  diff     - difference between successive approximations of the integral
%  d2r      - degree to radian conversion factor 57.29577951...
%  first    - first value of f(x)
%  fx       - value of f(x)
%  h        - interval width
%  Integral - numerical value of integral from trapezoidal rule
%  k,j      - integer counters
%  last     - last value of f(x)
%  m        - maximum number of intervals
%  n        - number of intervals
%  S        - array of integral values
%  sum      - sum of function values
%  x        - the variable
%
% References:
%  Williams, P. W., (1972), "Numerical Computation", Nelson, London.
%
%--------------------------------------------------------------------------

% Degree to radian conversion factor
d2r   = 180/pi;

fprintf('\n\nRomberg Integration Table for the integral of sec(x) for x = 45 degrees');

% Set array of values S(k,j)
m = 15;
S = zeros(m,m);
finish = 0;
for k = 1:m
    % set the number of intervals and the increment
```

```
    n = 2^k;
    h = 45/n;
    sum = 0;
    % evaluate the integral using the Trapezoidal Rule
    for x = 0:h:45
        fx = 1/cos(x/d2r);
        sum = sum+2*fx;
        if x == 0
            first = fx;
        end
        last = fx;
    end
    sum = sum-first-last;
    Integral = h/d2r/2*sum;
    S(k,1) = Integral;
    fprintf('\n%d %15.12f',k,S(k,1));
    % Use Richardson extrapolation
    for j = 2:k
        S(k,j) = 1/(4^(j-1)-1)*(4^(j-1)*S(k,j-1)-S(k-1,j-1));
        fprintf(' %15.12f',S(k,j));
        diff = abs(S(k,j-1)-S(k,j));
        if diff < 1e-12
            finish = 1;
            break;
        end
    end
    if finish == 1
        break;
    end
end
fprintf('\n\n');
```

MATLAB Command Window

```
>> help romberg_test

  This function cumputes the numerical value of the integral of sec(x)
  which is known to equal ln[tan(x/2+pi/4)].
  For x = 45 degrees the integral sec(x) = 0.881373587020.
  An integration table is produced that shows the convergence to the true
  value of the integral.


>> romberg_test


Romberg Integration Table for the integral of sec(x) for x = 45 degrees
1  0.899084147577
2  0.885885914440  0.881486503395
3  0.882507477613  0.881381332003  0.881374320577
4  0.881657432521  0.881374084157  0.881373600967  0.881373589544
5  0.881444571861  0.881373618307  0.881373587251  0.881373587033  0.881373587023
6  0.881391334699  0.881373588978  0.881373587023  0.881373587020  0.881373587020

>>
```

The output from the function *Romberg_ test.m* that is evaluating the integral

$$I = \int_{x=0^\circ}^{x=45^\circ} \sec\left(x\right) dx$$

is shown in the Romberg Integration Table and the elements are obtained as follows:

- For $k = 1$, there are $n = 2^k = 2$ intervals (or strips) of width $h$ where

$h = \dfrac{b-a}{n} = \dfrac{45^\circ - 0^\circ}{2} = 22.50^\circ$ and the integral $I \simeq \dfrac{h}{2}\left(f_0 + 2f_1 + f_2\right)$. The function $f\left(x\right) = \sec x = \dfrac{1}{\cos x}$ evaluated at $x = 0^\circ, 22.5^\circ, 45^\circ$ gives

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1.082392200 \\ f_2 &= 1.414213562 \end{aligned}$$

and

$$S_{1,1} = I = \frac{22.5}{2}\left(\frac{\pi}{180}\right)\left(1 + 2\left(1.082392200\right) + 1.414213562\right) = 0.899084148$$

- For $k = 2$, there are $n = 2^k = 4$ intervals (or strips) of width $h$ where

$h = \dfrac{b-a}{n} = \dfrac{45^\circ - 0^\circ}{4} = 11.25^\circ$ and the integral $I \simeq \dfrac{h}{2}\left(f_0 + 2f_1 + 2f_3 + f_4\right)$. The function $f\left(x\right) = \sec x = \dfrac{1}{\cos x}$ evaluated at $x = 0^\circ, 11.25^\circ, 22.5^\circ, 33.75^\circ, 45^\circ$ gives

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1.019591158 \\ f_2 &= 1.082392200 \\ f_3 &= 1.202689774 \\ f_4 &= 1.414213562 \end{aligned}$$

and

$$S_{2,1} = I = \frac{11.25}{2}\left(\frac{\pi}{180}\right)\left(1 + 2\left(1.019\ldots\right) + 2\left(1.082\ldots\right) + 2\left(1.202\ldots\right) + 1.414\ldots\right) = 0.885885914$$

The element $S_{2,2}$ is obtained from equation (60)

$$S_{2,2} = \frac{1}{4^1 - 1}\left(4^1 S_{2,1} - S_{1,1}\right) = \frac{1}{3}\left(4 \times 0.885885914 - 0.899084148\right) = 0.881486503$$

- For $k=3$, there are $n=2^k=8$ intervals (or strips) of width $h=5.625^\circ$ and the integral $I \simeq \frac{h}{2}\left(f_0+2f_1+2f_3+\cdots+2f_7+f_8\right)$. The function $f\left(x\right)=\sec x=\frac{1}{\cos x}$ evaluated at $x=0^\circ, 5.625^\circ, 11.25^\circ, \ldots, 39.375^\circ, 45^\circ$ gives

$$
\begin{aligned}
f_0 &= 1\\
f_1 &= 1.004838572\\
f_2 &= 1.019591158\\
&\vdots\\
f_7 &= 1.293643567\\
f_8 &= 1.414213562
\end{aligned}
$$

and

$$S_{3,1}=I=\frac{5.625}{2}\left(\frac{\pi}{180}\right)\left(1+2\left(1.004\ldots\right)+\cdots+2\left(1.293\ldots\right)+1.414\ldots\right)=0.882507478$$

The elements $S_{3,2}$ and $S_{3,3}$ are obtained from equation (60)

$$
\begin{aligned}
S_{3,2} &= \frac{1}{4^1-1}\left(4^1 S_{3,1}-S_{2,1}\right)=\frac{1}{3}\left(4\times 0.882507478-0.885885914\right) &&= 0.881381333\\
S_{3,3} &= \frac{1}{4^2-1}\left(4^2 S_{3,2}-S_{2,2}\right)=\frac{1}{15}\left(16\times 0.881381333-0.881486503\right) &&= 0.8813374322
\end{aligned}
$$

And so on for increasing values of $k$

Testing between successive values $S_{k,j-1}$ and $S_{k,j}$ can be used to determine when the iterative procedure is terminated.